ORBITS OF ORTHOGONAL AND SYMPLECTIC REPRESENTATIONS OF SYMMETRIC QUIVERS

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Abstract. When a quiver Q comes equipped with a certain contravariant involution, it becomes possible to define symmetric quivers along with orthogonal and symplectic representations. The notion of symmetric quivers was introduced in 2002 by Derksen and Weyman and naturally carries many parallels with the usual $GL(V_\Sigma)$ representations of a quiver. In this article, we first outline the definitions and the categorical framework for orthogonal and symplectic representations. Next, we prove that the codimension of the orbit of orthogonal or symplectic representations V of a symmetric quiver is equal not to $\dim \operatorname{Ext}(V,V)$ but to the dimension of a subspace invariant under a certain involution. Finally, we adapt results by Reineke to construct desingularizations of all orbit closures for orthogonal or symplectic representations of a finite type quiver.

1. Introduction - Motivation

In the recent article [8], Derksen and Weyman provide definitions for what they call generalized quivers associated to a reductive group G. In this paper, we briefly restate the notion of orthogonal and symplectic representations of symmetric quivers, provide categorical language for such representations and study a few properties of the orbits of representations when acted on by the appropriate classical Lie group $GL_Q(\alpha)^\sigma$. In particular, in Theorem 3.5, we prove that the codimension of the closure of the orbit O_V of $GL_Q(\alpha)^\sigma$ acting on a representation $V \in \operatorname{Rep}_K(Q,\alpha)^\sigma$, the space of orthogonal (resp. symplectic) representations, is:

$$\operatorname{codim} O_V = \dim \operatorname{Ext}(V,V)^{(-\varphi)}$$

where φ is a certain linear involution on the K-vector space $\operatorname{Ext}(V,V)$. This is an analogue of what is sometimes refered to as the Artin-Voight formula. Furthermore, in Theorem 4.8 we adapt recent work by Reineke to construct desingularizations of orbit closures of O_V in $\operatorname{Rep}_K(Q,\alpha)^\sigma$.

This paper is organized as follows. Section 2 provides the defitions for symmetric quivers and discusses the group actions on relevant affine spaces as well as categorical properties of orthogonal and symplectic representations. The main theorems of this article are in the following two sections: in Section 3 we prove the codimension formula for closures of orbits of orthogonal and symplectic representations, while in Section 4 we construct the resolution of singularities for these orbit closures. Finally, in Section 4, we apply the results in previous sections to study in detail orthogonal and symplectic representations of the symmetric equioriented A_n quiver, including a description of indecomposable objects, numerical formulas for codimensions of orbit closures and explicit desingularizations of the orbit closures.

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2. Symmteric Quivers and Their Representations

2.1. **Standard Quivers.** Let Q be a quiver, that is a quadruple $Q = (Q_0, Q_1, t, h)$ consisting of: 1) a discrete set Q_0 of elements we call vertices; 2) a set Q_1, the elements of which we call edges or arrows; and 3) two maps $t, h : Q_1 \longrightarrow Q_0$.

Fix a field K. A representation V of the quiver Q consists of vector space V_i for each vertex $i \in Q_0$ and linear maps $V(a) \in \mathrm{Hom}(V_{ta}, V_{ha})$ for all arrows $a \in Q_1$. A morphism between quiver representations V and W consists of a collection of homomorphisms $f_i : V_i \longrightarrow W_i$ such that for each arrow $a \in Q_1$, the following square commutes:

$$\begin{tikzcd} V_{ta} \arrow[r, "V(a)"] \arrow[d, "f_{ta}"'] & V_{ha} \arrow[d, "f_{ha}"] \\ W_{ta} \arrow[r, "W(a)"'] & W_{ha} \end{tikzcd} \tag{1}$$

Representations if the quiver Q along with morphisms define the category KQ-$\mathfrak{Mod}$ (modules over the path algebra KQ.) If one specifies a dimension vector $\alpha : Q_0 \to \mathbb{N}$ where $\alpha(i) = \dim V_i$, then the space of all quiver representations with the fixed dimension vector α is an affine space:

$$\mathrm{Rep}_K(Q, \alpha) = \bigoplus_{a \in Q_1} \mathrm{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)}) \tag{2}$$

The group $GL_Q(\alpha) = GL(\alpha(1)) \times \ldots \times GL(\alpha(n))$ acts naturally on $\mathrm{Rep}_K(Q, \alpha)$ by

$$((g_i)_{i \in Q_0}) \cdot V = (g_{ha} V(a) g_{ta}^{-1})_{a \in Q_1}$$

This action amounts to changing the bases in each of the vector spaces V_i.

Using the canonical projective resolution of quiver representations, we obtain the following exact sequence, called Ringel's canonical resolution:

$$0 \to \mathrm{Hom}_Q(V, W) \to \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i) \xrightarrow{d} \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, W_{ha}) \to \mathrm{Ext}_Q(V, W) \to 0 \tag{3}$$

where $d((f_i)_{i \in Q_0}) = (\psi_a \circ f_{ta} - f_{ha} \circ \phi_a)_{a \in Q_1}$. This map d should not seem so mysterious since we defined morphisms in $\mathrm{Hom}_Q(V, W)$ as elements of $\bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i)$ such that the square in (1) commutes for all arrows $a \in Q_1$, i.e. such that all $d((f_i)_{i \in Q_0}) = 0$ for $(f_i) \in \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i)$.

As a simple consequence of Ringel's resolution one deduces the well-known Artin-Voigt formula for the codimension of the orbit O_V of a quiver representation $V \in \mathrm{Rep}_K(Q, \alpha)$ under the action of $GL_Q(\alpha)$:

$$\operatorname{codim} O_V = \dim \mathrm{Ext}(V, V) \tag{4}$$

2.2. **Symmetric Quivers.** In this section, we recall the definitions of a symmetric quiver and its related symplectic or orthogonal representations as first stated in paper [8] and we refer the reader to that article for examples and proofs of the facts we shall only state.

Definition 2.1. A symmetric quiver (Q, σ) consists of an underlying quiver $Q = (Q_0, Q_1, h, t)$ and a bijection $\sigma : Q_0 \coprod Q_1 \longrightarrow Q_0 \coprod Q_1$ with $\sigma(Q_0) = Q_0$ and $\sigma(Q_1) = Q_1$ such that:

- $\forall a \in Q_1, h(\sigma(a)) = \sigma(t(a))$
- $\forall a \in Q_1, t(\sigma(a)) = \sigma(h(a))$
- if $\sigma(t(a)) = h(a)$ then $\sigma(a) = a$

As a simple mnemonic, we note that heads go to tails and vice versa. More important, we notice that the three conditions imposed on σ garantee that it is an involution on Q_0 and Q_1.

Let Q be a quiver and let $V \in \mathrm{Rep}_K(Q,\alpha)$ be a quiver representation. Define $V_\Sigma = \bigoplus_{i\in Q_0} V_i$ and call V_Σ the **total vector space** of V. We remark two immediate uses of the total vector space of quiver representations. First, if $\psi : V \longrightarrow V'$ is a morphism between two representations, then we can define $\psi_\Sigma \in \mathrm{Hom}(V_\Sigma, V'_\Sigma)$ such that for all $i \in Q_0$ and all $v \in V_i$, $\psi_\Sigma(v)$ is the canonical image of $\psi_i(v)$ in V'_Σ.

Secondly, we remark that for every $a \in Q_1$, we may view $V(a)$ as an element of $\mathrm{End}(V_\Sigma)$ that is zero except in the block that maps V_{ta} to V_{ha}. In particular, if the graph of the quiver Q is a tree, we may collect all the $V(a)$ that consitute a Q-representation into a single nilpotent element of $\mathrm{End}(V_\Sigma)$.

2.3. **Linear Algebra Reminders.** The definitions for representations of symmetric quivers rely on a simple linear algebra fact which we repeat here.

Let $\langle\cdot,\cdot\rangle$ be a nondegenerate bilinear form on a vector space W. Let $A : W \to W$ be a linear map. There exists a unique linear map $A^\star : W \to W$ such that $\langle Ax, y\rangle = \langle x, A^\star y\rangle$ for all $x, y \in W$. We call $A^\star$ the **adjoint** of A with respect to $\langle\cdot,\cdot\rangle$.

Proposition 2.2. *Let $\langle\cdot,\cdot\rangle$ be a non-degenerate bilinear form on a vector space W and let M be the matrix used to define it, i.e. $\langle v, w\rangle = v^t M w$ for all $v, w \in W$. Then the following hold:*

1. $A^\star = M^{-1}(A^t)M$.
2. *An element $g \in GL(W)$ which preserves $\langle\cdot,\cdot\rangle$ satisfies $g = (g^{-1})^\star$.*
3. *An element A in the Lie algebra of the Lie group which preserves $\langle\cdot,\cdot\rangle$ satisfies $A = -A^\star$.*

2.4. **Orthogonal and Symplectic Representations of Symmetric Quivers.** Consider a symmetric quiver Q along with a dimension vector $\alpha \in \mathbb{N}^{Q_0}$ that is symmetric under the involution, i.e. $\alpha(\sigma(i)) = \alpha(i)$ for all $i \in Q_0$. Let $\langle\cdot,\cdot\rangle$ be a nondegenerate scalar product on the total vector space $K^\alpha = \bigoplus_{i\in Q_0} K^{\alpha(i)}$ such that

$$\langle\cdot,\cdot\rangle|_{K^\alpha_p \times K^\alpha_q} = 0 \qquad \text{unless } q = \sigma(p) \tag{5}$$

In light of (3) in Proposition 2.2, we make the following defintion.

Definition 2.3. With respect to a form with the above property, for all $V \in \mathrm{Rep}_K(Q,\alpha)$ we can define an adjoint object $V^\star \in \mathrm{Rep}_K(Q,\alpha)$ by:

- $(V^\star)_i = (V_{\sigma(i)})^*$, the dual vector space
- $(V^\star)(a) = -(V(\sigma(a)))^\star$, the *negative* of the adjoint of $V(a)$ considered as an element of $\mathrm{End}(K^\alpha)$.

As a point of notation, we will also sometimes write $\sigma \cdot V$ instead of $V^\star$ to express this operator.

Definition 2.4. Let Q be a symmetric quiver, α a symmetric dimension vector and $\langle\cdot,\cdot\rangle$ be a nondegenerate bilinear form on the total vector space K^α, satisfying property (5). Let $\mathrm{Rep}_K(Q,\alpha)^\sigma$ be the space of all Q-representations V such that $\sigma \cdot V \equiv V^\star = V$, i.e. the self-dual objects with respect to $\langle\cdot,\cdot\rangle$. We call an element of $\mathrm{Rep}_K(Q,\alpha)^\sigma$ a symplectic (respectively, orthogonal) quiver representation when $\langle\cdot,\cdot\rangle$ is skew-symmetric (resp. symmetric).

2.5. **Group Actions.** By Proposition 2.2, the subgroup G^σ which stabilizes $\mathrm{Rep}_K(Q,\alpha)^\sigma$ consists of elements $g \in G$ such that $g = (g^{-1})^\star$ where we view g as an element of $\mathrm{End}(V_\Sigma)$. Now for $g \in G = GL_Q(\alpha)$, g is given as a collection $g = (g_i)_{i\in Q_0}$. If we consider each g_i as an element of $\mathrm{End}(V_\Sigma)$, then we can write

$$\sigma \cdot ((g_i)_{i\in Q_0}) = ((g^{-1}_{\sigma(i)})^\star)_{i\in Q_0}$$

In order to get a more explicit description of G^σ we must partition the vertices of the symmetric quiver (Q, σ).

For a symmetric quiver without oriented cycles, call the Q_0^σ (Q_1^σ resp.) the set of vertices (arrows resp.) fixed by σ. There exists partitions

$$\begin{aligned} Q_0 &= Q_0^+ \cup Q_0^\sigma \cup Q_0^- \\ Q_1 &= Q_1^+ \cup Q_1^\sigma \cup Q_1^- \end{aligned} \tag{6}$$

such that $Q_0^- = \sigma(Q_0^+)$ and $Q_1^- = \sigma(Q_1^+)$ and satisfying:

- $\forall a \in Q_1^+$, either $\{ta, ha\} \subset Q_0^+$ or one of the $\{ta, ha\}$ is in Q_0^+ while the other is in Q_0^σ.
- $\forall x \in Q_0^+$, if $a \in Q_1$ with $ta = x$ or $ha = x$, then $a \in Q_1^+ \cup Q_1^\sigma$

It is easy to see how to construct such partitions with examples so we leave the details to the interested reader.

Proposition 2.5. *Let Q be a symmetric quiver and α a symmetric dimension vector. Partition Q_0 and Q_1 into Q_i^+, Q_i^σ and Q_i^+ (where $i = 0, 1$) according to Lemma 6. The following hold:*

- *If the action of σ is symplectic,*

$$\begin{aligned} Rep_K(Q, \alpha)^\sigma &\cong \bigoplus_{a \in Q_1^+} \mathrm{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)}) \oplus \bigoplus_{a \in Q_1^\sigma} Sym^2(K^{\alpha(ta)}) \\ GL_Q(\alpha)^\sigma &= \prod_{x \in Q_0^+} GL(K^{\alpha(x)}) \times \prod_{x \in Q_0^\sigma} Sp(K^{\alpha(x)}) \end{aligned}$$

- *If the action of σ is orthogonal,*

$$\begin{aligned} Rep_K(Q, \alpha)^\sigma &\cong \bigoplus_{a \in Q_1^+} \mathrm{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)}) \oplus \bigoplus_{a \in Q_1^\sigma} \bigwedge^2 (K^{\alpha(ta)}) \\ GL_Q(\alpha)^\sigma &= \prod_{x \in Q_0^+} GL(K^{\alpha(x)}) \times \prod_{x \in Q_0^\sigma} O(K^{\alpha(x)}) \end{aligned}$$

Proof. See section 2 of [8]. □

2.6. **The Categories of Orthogonal/Symplectic Representations of a Symmetric Quiver.**

In this subsection, we briefly outline some basic properties about orthogonal and symplectic representations of symmetric quivers and refer the reader to [8] for proofs and more explanation.

2.6.1. *Objects and Morphisms.* From the presentation in the previous section, it is clear that given a symmetric quiver Q and a symmetric dimension vector α, we called a representation $V \in \mathrm{Rep}_K(Q, \alpha)$ symplectic or orthogonal with respect to some nondegenerate bilinear form satisfying property (5). Consequently, the bilinear form must be part of the data included in defining objects.

Definition 2.6. Let Q be a symmetric quiver. The category KQ-$\mathfrak{Mod}_o$ (resp. KQ-$\mathfrak{Mod}_s$) of orthogonal (resp. symplectic) quiver representations has objects which consist of a pair $(V, \langle \cdot, \cdot \rangle_V)$ where $\mathbf{dim}\, V$ is a symmetric dimension vector and V is a Q°-representation together with an orthogonal (resp. symplectic) non-degenerate bilinear form $\langle \cdot, \cdot \rangle_V$ satisfying property (5) such that $V \in \mathrm{Rep}_K(Q, \mathbf{dim}\, V)^\sigma$. The morphisms $f : V \to W$ between objects in KQ-$\mathfrak{Mod}_o$ (resp. KQ-$\mathfrak{Mod}_s$) are the morphisms between V and W considered as KQ-modules.

By definition of the categories, we see that for any two orthogonal (resp. symplectic) quiver representations $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$,

$$\mathrm{Mor}((V, \langle\cdot,\cdot\rangle_V), (W, \langle\cdot,\cdot\rangle_W)) \cong \mathrm{Mor}(V, W)$$

Therefore $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ are **isomorphic** as orthogonal (resp. symplectic) quiver representations if and only if V and W are isomorphic in KQ-$\mathfrak{Mod}$. Furhtermore, the functor $F_o : KQ\text{-}\mathfrak{Mod}_o \to KQ\text{-}\mathfrak{Mod}$ (resp. F_s) which forgets the form $\langle\cdot,\cdot\rangle$, exhibits KQ-$\mathfrak{Mod}_o$ (resp. KQ-$\mathfrak{Mod}_s$) as a full subcategory of the usual KQ-modules. The following proposition underscores states the desired result that isomorphism classes correspond to orbits of $GL_Q(\alpha)^\sigma$ acting on $\mathrm{Rep}_K(Q, \alpha)^\sigma$.

Proposition 2.7. *Let Q be a symmetric quiver. Two orthogonal (resp. symplectic) representations $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ in KQ-$\mathfrak{Mod}_o$ (resp. KQ-$\mathfrak{Mod}_s$) have isomorphic underlying representations if and only if* $\mathbf{dim}\, V = \mathbf{dim}\, W := \alpha$ *and V and W are in the same $GL_Q(\alpha)^\sigma$ orbit in $Rep_K(Q, \alpha)^\sigma$.*

Proof. Follows immediately from Theorem 2.6 in [8]. ☐

2.6.2. *Adjoint Morphism.*

Definition 2.8. Let (Q, σ) be a symmetric quiver and α a symmetric dimension vector. Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be two orthogonal (resp. symplectic) representations of (Q, σ) and let $f : V \to W$ be a morphism between them. We define an **adjoint morphism** $f^\star : W^\star \to V^\star$ as being the unique morphism such that

$$\langle f(v), w\rangle_W = \langle v, f^\star(w)\rangle_V \qquad \text{for all } v \in V_\Sigma \text{ and } w \in W_\Sigma \tag{7}$$

Since $V^\star = V$ and $W^\star = W$, the adjoint morphism is in fact a map $f^\star : (W, \langle\cdot,\cdot\rangle_W) \to (V, \langle\cdot,\cdot\rangle_V)$.

Note that with any choice of bases in V_i and W_i for all $i \in Q_0$, one can write a morphism of quiver representations $f : V \to W$ as a block diagonal matrix $f_\Sigma \in \mathrm{Hom}(V_\Sigma, W_\Sigma)$. If with this choice of bases, the bilinear forms $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$ have defining matrices M_V and M_W, then

$$(f^\star)_\Sigma = M_V^{-1} f_\Sigma^t M_W$$

2.6.3. *Indecomposable Representations.*

Definition 2.9. If $(V_1, \langle\cdot,\cdot\rangle_1)$ and $(V_2, \langle\cdot,\cdot\rangle_2)$ are two orthogonal (resp. symplectic) representations of a symmetric quiver Q, then the direct sum is given by $(W, \langle\cdot,\cdot\rangle_W)$ where $W = V_1 \oplus V_2$ in KQ-$\mathfrak{Mod}$ and the scalar product $\langle\cdot,\cdot\rangle_W$ on W_Σ is the sum $\langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$.

One immediately deduces the following useful fact.

Proposition 2.10. *The categories KQ-$\mathfrak{Mod}_o$ and KQ-$\mathfrak{Mod}_s$ are additive categories (though not in general abelian).*

The following propositions relate the notion of indecomposability in KQ-$\mathfrak{Mod}_o$ (resp. KQ-$\mathfrak{Mod}_s$) to that in KQ-$\mathfrak{Mod}$.

Proposition 2.11 (Proposition 2.7 in [8])**.** *If $(V, \langle\cdot,\cdot\rangle)$ is an indecomposable orthogonal (resp. symplectic) representation of the symmetric quiver Q, then as a representation of the underlying quiver Q°, V is indecomposable or isomorphic to the direct sum of W and $W^\star$, where W is an indecomposable representation of KQ-$\mathfrak{Mod}$.*

Proposition 2.12. *If $(V, \langle\cdot,\cdot\rangle)$ is an indecomposable orthogonal or symplectic representation quiver, then three possibilities occur for V as a representation of the underlying quiver Q°:*

1. *V is indecomposable;*

(2) *$V = W \oplus W^\star$ and W is indecomposable such that $W \cong W^\star$ (V* ramifies*);*
(3) *$V = W \oplus W^\star$ and W is indecomposable such that W and $W^\star$ are not isomorphic (V* splits*).*

Proposition 2.13. *Let Q be a symmetric quiver and V an indecomposable KQ-module. Then exactly one of the following two possibilities occur:*

(1) *there exists a symmetric (resp. skew-symmetric) nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on V such that $(V, \langle \cdot, \cdot \rangle)$ is an indecomposable orthogonal (resp. symplectic) representation of Q;*
(2) *there exists an indecomposable orthogonal (resp. symplectic) representation $(\tilde{V}, \langle \cdot, \cdot \rangle)$ such that V is a direct summand of $\tilde{V}$ in KQ-$\mathfrak{Mod}$ and furthermore that $\tilde{V} = V \oplus V^\star$, where the adjoint is taken with respect to $\langle \cdot, \cdot \rangle$.*

Proposition 2.14. *A symmetric quiver (Q, σ) is of finite type if and only if the underlying quiver Q is of finite type.*

Corollary 2.15. *Any connected, symmetric quiver of finite type has an underlying graph of type A_n.*

We illustrate the propositions with two simple examples, one in which the quiver has only one fixed arrow, and one in which the quiver has one fixed vertex.

Example 2.16. Consider the symmetric quiver A_2: $1 \xrightarrow{a} 2$. The indecomposable KQ°-modules are:

$$E_{11} : K \xrightarrow{0} 0 \qquad E_{12} : K \xrightarrow{\mathrm{id}} K \qquad E_{22} : 0 \xrightarrow{0} K$$

Let us first consider symplectic representations of A_2. Using Proposition 2.11, it is easy to see that the following symplectic representations are indecomposable

$$E_{11} \oplus E_{22} : K \xrightarrow{0} K \qquad E_{12} : K \xrightarrow{\mathrm{id}} K$$

where each has the skew-symmetric form $\langle v, \lambda \rangle = -\lambda(v)$ on $V_1 \oplus V_2 = V_1 \oplus V_1^*$. Furthermore, it is easy to see that there can be no other symplectic indecomposables.

As for orthogonal representations of A_2, the difference comes from the fact that the linear map $V(a)$ must be in a space that is isomorphic as a vector space to the space of skew-symmetric matrices. Hence the indecomposables orthogonal modules are:

$$E_{11} \oplus E_{22} : K \xrightarrow{0} K \qquad E_{12}^2 : K \xrightarrow{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} K$$

Example 2.17. Consider now the symmetric quiver A_3: $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ in which only the middle vertex is fixed the involution σ. The indecomposable KQ°-modules are:

$$E_{11} : K \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad E_{22} : 0 \xrightarrow{0} K \xrightarrow{0} 0 \qquad E_{33} : 0 \xrightarrow{0} 0 \xrightarrow{0} K$$

$$E_{12} : K \xrightarrow{\mathrm{id}} K \xrightarrow{0} 0 \qquad E_{23} : 0 \xrightarrow{0} K \xrightarrow{\mathrm{id}} K \qquad E_{13} : K \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K$$

This time, let us consider the orthogonal representations of A_3 first. Using Proposition 2.11, we see that the following orthogonal representations are indecomposable where we have described the from $\langle \cdot, \cdot \rangle$ by a defining matrix M on $V_1 \oplus V_2 \oplus V_3$.

$$E_{11} \oplus E_{33} : K \xrightarrow{0} 0 \xrightarrow{0} K \qquad \text{where } M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$E_{12} \oplus E_{23} : K \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} K^2 \xrightarrow{(0\ -1)} K \qquad \text{where } M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{13}: \ K \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K \qquad \text{where } M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$E_{22}: \ 0 \xrightarrow{0} K \xrightarrow{0} 0 \qquad \text{where } M = (1)$$

Also by Proposition 2.11, we see that this exhauts the list of all possible orthogonoal indecomposable modules.

As for symplectic representations $(V, \langle \cdot, \cdot \rangle)$ of A_3, we must remember that $\dim V_2$ must be even. Again using Proposition 2.11, one can check that the following are all the indecomposable symplectic representations.

$$E_{11} \oplus E_{33}: \ K \xrightarrow{0} 0 \xrightarrow{0} K \qquad \text{where } M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$E_{12} \oplus E_{23}: \ K \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} K^2 \xrightarrow{(0\ 1)} K \qquad \text{where } M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{13}^2: \ K^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} K^2 \xrightarrow{\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} K^2 \qquad \text{where } M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{22}^2: \ 0 \xrightarrow{0} K^2 \xrightarrow{0} 0 \qquad \text{where } M = \mathrm{id}_{V_2}$$

Remark 2.1. In much of what follows, it turns out we can provide formulas which hold simultaneously for orthogonal representations and symplectic representations if we continue to employ the parameter

$$\varepsilon = \begin{cases} 1 & \text{if the form } \langle \cdot, \cdot \rangle \text{ is orthogonal} \\ -1 & \text{if the form } \langle \cdot, \cdot \rangle \text{ is symplectic} \end{cases} \tag{8}$$

3. Codimension of Orbits of Orthogonal or Symplectic Representations

In this section, we fix a symmetric quiver (Q, σ) and symmetric dimension vector α and focus our attention on the orbits of an orthogonal (resp. symplectic) representations in $\mathrm{Rep}_K(Q, \alpha)^\sigma$. To be more specific, we choose an orthogonal (resp. symplectic) representation $(V, \langle \cdot, \cdot \rangle) \in \mathrm{Rep}_K(Q, \alpha)^\sigma$ and investigate its orbit O_V under the action of $GL_Q(\alpha)^\sigma$. We establish a codimension formula similar to Equation 4 which states that for an orbit O_V of a representation of standard quiver Q, one has

$$\operatorname{codim} O_V = \dim \operatorname{Ext}_Q(V, V)$$

There exists a similar formula for symmetric quivers but we must first understand how the adjoint operator $\sigma : V \to V^\star$ acts on vector spaces $\operatorname{Hom}_Q(V, W)$ and $\operatorname{Ext}_Q(V, W)$. We then illustrate the formula by performing explicit calculations with symmetric A_2 and A_3 quivers.

3.1. **An Algebraic Codimension Formula.**

3.1.1. *Functoriality of σ.* Let us fix a symmetric quiver (Q,σ), a symmetric dimension vector α and a nondegenerate orthogonal or symplectic bilinear form $\langle\cdot,\cdot\rangle$ on the total vector space K^α satisfying property 5:

$$\langle\cdot,\cdot\rangle|_{K_p^\alpha\times K_q^\alpha}=0\qquad\text{unless } q=\sigma(p)$$

We will consider all representations V as orthogonal or symplectic with respect to this form.

As we saw earlier, the involution σ acts on $\mathrm{Rep}_K(Q,\alpha)$ as an adjoint operator duality, i.e. $\sigma\cdot V=V^\star$ where the adjoint is of course with respect to $\langle\cdot,\cdot\rangle$ on $K^\alpha=V_\Sigma$. If one views $\mathrm{Rep}_K(Q,\alpha)$ as a vector space, then one may interpret the action of σ as an involutive automorphism. On the other hand, if one views $\mathrm{Rep}_K(Q,\alpha)$ as a subcategory of KQ-$\mathfrak{Mod}$, then σ acts a contravariant invertible functor from $\mathrm{Rep}_K(Q,\alpha)$ to itself. In particular, as a functor σ sends $V\in\mathrm{Rep}_K(Q,\alpha)$ to its adjoint object $V^\star$ and sends a morphism $f:V\to W$ to its adjoint morphism $f^\star:W^\star\to V^\star$ defined in Equation 7.

Proposition 3.1. *Let $V,W\in Rep_K(Q,\alpha)$ with the bilinear form $\langle\cdot,\cdot\rangle$ defined as above. The involution σ of the symmetric quiver Q induces natural isomorphisms*

$$F_\sigma:\mathrm{Hom}_Q(V,W)\longrightarrow\mathrm{Hom}_Q(W^\star,V^\star)\qquad\textit{and}\qquad G_\sigma:\mathrm{Ext}_Q(V,W)\longrightarrow\mathrm{Ext}_Q(W^\star,V^\star)$$

Proof. The first isomorphism is trivial since we simply define F_σ by $F_\sigma(f)=f^\star$, the adjoint morphism of f.

The second homomorphism, G_σ is slightly less trivial but we may define it as follows. If $e\in\mathrm{Ext}(V,W)$, has a representative given by the exact sequence

$$0\longrightarrow W\xrightarrow{f}E\xrightarrow{g}V\longrightarrow 0$$

Note that as total vector spaces, $E_\Sigma\cong V_\Sigma\oplus W_\Sigma$. Consequently, we can define a natural nondegenerate bilinear form

$$\langle\cdot,\cdot\rangle_E=\langle\cdot,\cdot\rangle_V+\langle\cdot,\cdot\rangle_W$$

on E. Then we define $G_\sigma(e)$ as the equivalence class in $\mathrm{Ext}(W^\star,V^\star)$ corresponding to the exact sequence

$$0\longrightarrow V^\star\xrightarrow{g^\star}E^\star\xrightarrow{f^\star}W^\star\longrightarrow 0$$

where the connecting maps are the adjoints of the maps in the previous sequence with respect to the appropriate quiver representations. One then easily checks that G_σ is well-defined. □

For the sake of calculations, we note that *, Hom_Q and Ext_Q are all additive functors so we can understand how F_σ and G_σ act on $\mathrm{Hom}_Q(V,W)$ or on $\mathrm{Ext}_Q(V,W)$ by expressing V and W as direct sums of indecomposable representations of the underlying quiver Q^o and then determining how F_σ acts on $\mathrm{Hom}_Q(M_x,M_y)$ or $\mathrm{Ext}_Q(M_x,M_y)$ for pairs of indecomposables (M_x,M_y). We will use this tactic in later examples.

Consider now the case when $V=W$ and $(V,\langle\cdot,\cdot\rangle)$ is either an orthogonal or symplectic representation of a symmetric quiver Q. By definition, $V=V^\star$ so in particular $\mathrm{Hom}_Q(V^\star,V^\star)=\mathrm{Hom}_Q(V,V)$ and $\mathrm{Ext}_Q(V^\star,V^\star)=\mathrm{Ext}_Q(V,V)$. Consequently, by Proposition 3.1, σ induces involutive endomorphisms on all the vector spaces in Ringel's resolution (3) namely:

$$\mathrm{Hom}_Q(V,V)\qquad\bigoplus_{i\in Q_0}\mathrm{Hom}(V_i,V_i)\qquad\bigoplus_{a\in Q_1}\mathrm{Hom}(V_{ta},V_{ha})\quad\text{and}\quad\mathrm{Ext}_Q(V,V)$$

For the sake of simplicity, we will generically refer to these linear maps as φ. We describe the behavior of φ on each of the vector spaces in (3):

- On $\mathrm{Hom}_Q(V,V)$, φ is induced by F_σ;
- For $(f_i)\in\bigoplus\mathrm{Hom}(V_i,V_i)$, we get $\varphi(f_i)_{i\in Q_0}=(f^\star_{\sigma(i)})_{i\in Q_0}$;
- For $(f_a)\in\bigoplus\mathrm{Hom}(V_a,V_a)$, we get $\varphi(f_a)_{a\in Q_1}=(f^\star_{\sigma(a)})_{a\in Q_1}$;

- On $\mathrm{Ext}_Q(V,V)$, φ is induced by G_σ.

In the following, if T is a vector space that appears in the exact sequence (3), we denote by T^φ (resp. $T^{(-\varphi)}$ the maximal φ-invariant (resp. $(-\varphi)$-invariant) subspace of T.

3.1.2. *Codimension for Symmetric Quiver Representations.* We introduce two simple lemmas about the action of φ on the vector spaces in the Ringel canonical resolution.

Lemma 3.2. *Let V and W be vector spaces possessing involutions ϕ_1 and ϕ_2 respectively. Let $L : V \to W$ be a linear map with $L \circ \phi_2 = \phi_1 \circ L$. Then L is block diagonal with respect to the subspace decomposition $V = V^{\phi_1} \oplus V^{(-\phi_1)}$ and $W = W^{\phi_2} \oplus W^{(-\phi_2)}$.*

Proof. First, we claim that for any vector space T and any involution $\phi : T \to T$, ϕ is diagonalizable. Suppose the contrary, that there exists an involution $\phi : T \to T$ that is not diagonalizable. Then from basic linearly algebra we know there exist linearly independent vectors $v_1, v_2 \in T$ such that

$$\phi(v_1) = \lambda v_1 \qquad \text{and} \qquad \phi(v_2) = \lambda v_2 + v_1$$

for some $\lambda \in K$. Then

$$\phi^2(v_2)\lambda(\lambda v_2 + v_1) + \lambda v_1 = v_2$$

and hence since the vectors are linearly independent, one has the system of equations

$$\begin{cases} 2\lambda = 0 \\ \lambda^2 - 1 = 0 \end{cases}$$

Since this system has no solutions, we have proved the claim.

More can be said however. Since ϕ is an involution, then $\phi^2 = I$ and hence $(\phi - I)(\phi + I) = 0$ and we notice the well-known fact that involutions can only have eigenvalues in $\{1, -1\}$. Consequently, since T^ϕ (resp. $T^{-\phi}$) are the eigenspaces for the eigenvalues 1 and -1, the vector space T decomposes into the direct sum $T = T^\phi \oplus T^{-\phi}$.

Having established these simple facts about linear involutions, the lemma is easy. If $v \in V^{\phi_1}$ then $L(v) = \phi_2(L(v))$. The same holds for the involutions $-\phi_1$ and $-\phi_2$ and the lemma follows. □

Lemma 3.3. *Let (Q, σ) be a symmetric quiver without oriented cycles, α a symmetric dimmension vector and $\langle \cdot, \cdot \rangle$ a nondegenerate symmetric (resp. skew-symmetric) bilinear form on K^α satisfying property 5. Generically call φ the involutive linear map induced by σ on each of the K-vector spaces in Ringel's resolution. For any orthogonal (resp. symplectic) representation $V \in Rep_K(Q, \alpha)^\sigma$, the following diagram is commutative:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_Q(V,V) \arrow[r] \arrow[d, "\varphi"] & \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, V_i) \arrow[r, "d"] \arrow[d, "\varphi"] & \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha}) \arrow[r] \arrow[d, "-\varphi"] & \mathrm{Ext}_Q(V,V) \arrow[r] \arrow[d, "-\varphi"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_Q(V,V) \arrow[r] & \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, V_i) \arrow[r, "d"] & \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha}) \arrow[r] & \mathrm{Ext}_Q(V,V) \arrow[r] & 0
\end{tikzcd}$$

Proof. The first square is commutative by definition: the involution on $\mathrm{Hom}(V,V)$ is induced from its injection into $\bigoplus \mathrm{Hom}(V_i, V_i)$.

For the middle square, let $(f_i) \in \bigoplus \mathrm{Hom}(V_i, V_i)$. Its image in $\bigoplus \mathrm{Hom}(V_{ta}, V_{ha})$ via the map d is the sequence $(V(a) \circ f_{ta} - f_{ha} \circ V(a))_{a \in Q_1}$. Thus

$$\begin{aligned}(-\varphi \circ d)(f_i) &= -(V(a) \circ f_{ta} - f_{ha} \circ V(a))^{\star} \\ &= -\big((V(a) \circ f_{ta})^{\star} - (f_{ha} \circ V(a))^{\star}\big) \\ &= -((f_{ta})^{\star} \circ V(a)^{\star} - V(a)^{\star} \circ (f_{ha})^{\star}) \\ &= (-(f^{\star})_{\sigma(ta)} \circ V(\sigma(a)) + V(\sigma(a)) \circ (f^{\star})_{\sigma(ha)}) \\ &= d \circ \varphi(f_i)\end{aligned}$$

Finally, the argument for the third square is dual to the argument for the first commutative square. ☐

As an immediate corollary, we obtain:

Proposition 3.4. *The following sequence is exact:*

$$0 \longrightarrow \mathrm{Hom}_Q(V,V)^{\varphi} \longrightarrow \Big(\bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, V_i)\Big)^{\varphi} \longrightarrow \Big(\bigoplus_{\alpha \in Q_1} \mathrm{Hom}(V_{t\alpha}, V_{h\alpha})\Big)^{(-\varphi)} \longrightarrow \mathrm{Ext}_Q(V,V)^{(-\varphi)} \longrightarrow 0 \tag{9}$$

Proof. We combine the previous two lemmas and note that Ringel's resolution (3) splits into two block diagonal sequences, one of which being that described in the statement of the proposition. ☐

We are now in a position to prove the following theorem which calculates the codimension for orbits of representations of symmetric quivers in $\mathrm{Rep}_K(Q,\alpha)^{\sigma}$.

Theorem 3.5. *Let (Q,σ) be a symmetric quiver without oriented cycles, α a symmetric dimmension vector and $\langle \cdot, \cdot \rangle$ a nondegenerate symmetric (resp. skew-symmetric) bilinear form on K^{α} satisfying property 5. Consider an orthogonal (resp. symplectic) representation $V \in Rep_K(Q,\alpha)^{\sigma}$. Call O_V the $GL(\alpha)^{\sigma}$ orbit of V in $Rep_K(Q,\alpha)^{\sigma}$ and let φ be the involutive linear map on* Ext(V,V) *induced by σ. Then*

$$\mathrm{codim}(O_V) = \dim \Big(\mathrm{Ext}(V,V)^{(-\varphi)}\Big) \tag{10}$$

Proof. Let us denote $G = GL_Q(\alpha)$.

Let $V \in \mathrm{Rep}_K(Q,\alpha)$ be a usual quiver representation, ignoring symmetry. Then any element g in the stabilizer G_V defines an element of $\mathrm{Hom}(V,V)$. Conversely, if $f \in \mathrm{Hom}(V,V)$ with each f_i invertible, then $f \in G_V$. Hence, G_V is homeomorphic to an open subset of $\mathrm{Hom}(V,V)$, and in particular they have the same dimension.

Next, consider the action of σ on the subgroup $G_V \leq G$. We define a subgroup of G_V invariant under the automorphism induced by σ, namely $(G_V)^{\sigma} = \{g \in G_V \,|\, (g^{-1})^{\star} = g\}$. Then $(G_V)^{\sigma} = G_V \cap G^{\sigma}$. Thus, if $V \in \mathrm{Rep}_K(Q,\alpha)^{\sigma}$, then $(G^{\sigma})_V \cong (G_V)^{\sigma}$. We then deduce that $\dim \mathrm{Hom}(V,V)^{\varphi} = \dim (G^{\sigma})_V$.

Taking a dimension count on the exact sequence in Proposition 3.4, we get:

$$\dim \mathrm{Hom}(V,V)^{\varphi} - \dim \Big(\bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, V_i)\Big)^{\varphi} + \dim \Big(\bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha})\Big)^{(-\varphi)} - \dim \mathrm{Ext}(V,V)^{(-\varphi)} = 0 \tag{11}$$

However, we easily see the following:

$$\dim\Big(\bigoplus_{i\in Q_0}\mathrm{Hom}(V_i,V_i)\Big)^{\varphi}=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{i:\sigma(i)\neq i}(\dim V_i)^2+\sum_{i:\sigma(i)=i}\binom{\dim V_i}{2} & \text{if } \varepsilon=-1,\\ \dfrac{1}{2}\displaystyle\sum_{i:\sigma(i)\neq i}(\dim V_i)^2+\sum_{i:\sigma(i)=i}\binom{\dim V_i+1}{2} & \text{if } \varepsilon=1.\end{cases}$$

$$\dim\Big(\bigoplus_{a\in Q_1}\mathrm{Hom}(V_{ta},V_{ha})\Big)^{(-\varphi)}=$$

$$=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{a:\sigma(a)\neq a}\dim\mathrm{Hom}(V_{ta},V_{ha})+\sum_{a:\sigma(ha)=a}\binom{\dim V_{ta}+1}{2} & \text{if } \varepsilon=-1,\\ \dfrac{1}{2}\displaystyle\sum_{a:\sigma(a)\neq a}\dim\mathrm{Hom}(V_{ta},V_{ha})+\sum_{a:\sigma(a)=a}\binom{\dim V_{ta}}{2} & \text{if } \varepsilon=-1.\end{cases}$$

Hence, comparing the dimensions of these two identities with the results of Proposition 2.5, we can rewrite equation (11) as:

$$\big(\dim G^{\sigma}-\dim O_V\big)-\dim G^{\sigma}+\dim\mathrm{Rep}_K(Q,\alpha)^{\sigma}-\dim\mathrm{Ext}(V,V)^{(-\varphi)}=0$$

Thus we conclude that

$$\mathrm{codim}(O_V)=\dim\mathrm{Ext}(V,V)^{(-\varphi)}$$

□

In the next section, we present the simple examples of symmetric A_2 and A_3 quivers in which we construct adhoc desingularizations of orbit closures. We provide explicit descriptions of $\mathrm{Ext}_Q(V,V)$ and show how to calculate $\dim\mathrm{Ext}(V,V)^{(-\varphi)}$ and therefore verify Theorem 3.5 in these particular cases.

3.2. **Explicit calculations for symmetric A_2 and A_3.**

In both these examples, we can write formulas that simultaneously describe orthogonal and symplectic cases when we use the parameter ε where

$$\varepsilon=\begin{cases}1 & \text{orthogonal case}\\ -1 & \text{symplectic case}\end{cases}$$

3.2.1. *Codimension of Symmetric Representations of A_2.* The simplest symmetric quiver we can study is A_2. The general symmetric representation is given by:

$$V_1\xrightarrow{V(a)}V_1^*$$

where $\mathrm{rank}\,V(a)=r$ and $\dim V_1=n$. We can decompose this quiver representation, as a representation of the underlying quiver, into its indecomposable parts by $V=E_1^{s_1}\oplus E_2^{s_2}\oplus E_{1,2}^{r}$ where:

$$E_{11}=K\longrightarrow 0\qquad E_{22}=0\longrightarrow K\qquad E_{12}=K\xrightarrow{id}K$$

If V is an orthogonal or a symplectic representation, then $s_1=s_2$. The only non-split exact sequence we can create with just these three indecomposable modules in KQ-$\mathfrak{Mod}$ is the following:

$$0\longrightarrow E_{22}\longrightarrow E_{12}\longrightarrow E_{11}\longrightarrow 0$$

Thus, $\mathrm{Ext}(V,V)=\mathrm{Ext}(E_{11}^{s_1},E_{22}^{s_2})$. By equation (3), we know that

$$\mathrm{Ext}(V,V)=\mathrm{coker}\Big(d:\mathrm{End}(K^{s_1})\oplus\mathrm{End}(K^{s_1})\longrightarrow\mathrm{Hom}(K^{s_1},K^{s_1})\Big)$$

where in this particular case we have $d = 0$. Thus $\operatorname{Ext}(V, V) = \operatorname{Hom}(K^{s_1}, K^{s_1})$. Furthermore, the action of the linear map φ is given by $-\varphi(A) = -A^\star$. But $n = r + s_1$ so $\dim \operatorname{Ext}(V, V)^{(-\varphi)} = \frac{1}{2}(n-r)(n-r-1)$ when the action of σ is orthogonal and $\dim \operatorname{Ext}(V, V)^{(-\varphi)} = \frac{1}{2}(n-r)(n-r+1)$ when the action of σ is symplectic. Combining the two results into one formula, we can write $\dim \operatorname{Ext}(V, V)^{(-\varphi)} = \frac{1}{2}(n-r)(n-r-\varepsilon)$. We now find the codimension of the orbit $G^\sigma \cdot V$ in both the symplectic case and the orthogonal case.

To verify Theorem 3.5 we need to calculate the codimension of the orbit O_V of an orthogonal or symplectic representation $(V, \langle, \cdot, \cdot\rangle)$ under the action of $GL_Q((n, n))^\sigma$ in $\operatorname{Rep}_K(Q, \alpha)^\sigma$.

Symplectic Case: We assume that the action of σ on the quiver representation V is symplectic. There exists a form $\langle, \cdot, \cdot\rangle$ such that $V(a) \in \operatorname{Sym}^2(K^n)$ and $V(a)$ is of rank r. Since $V(a)$ is self dual, it essentially maps an r-dimensional subspace onto its dual. Given the subspace, the choice of the components of such a map gives us dimension $\frac{1}{2}r(r+1)$, whereas the choice of a subspace in $\mathbb{G}_r(K^n)$ contributes the dimension of the Grassmannian of r planes in K^n, namely $r(n-r)$.

More specifically, the orbit O_V has a desingularization $\pi : \mathbb{IG}(r, n) \to O_V$ where $\mathbb{IG}(r, n)$ is the symplectic isotropic Grassmanian. Thus we get:

$$\begin{aligned}
\operatorname{codim} O_V &= \frac{n(n+1)}{2} - r(n-r) - \frac{r(r+1)}{2} \\
&= \frac{(n-r)(n-r+1)}{2} \\
&= \dim \operatorname{Ext}_Q(V, V)^{(-\varphi)}
\end{aligned}$$

Orthogonal Case: The orthogonal case is similar except that we view $V(a)$ as an element of $\wedge^2(K^n)$. We get $\dim O_V = r(n-r) + \frac{1}{2}r(r-1)$ and hence:

$$\begin{aligned}
\operatorname{codim} O_V &= \frac{n(n-1)}{2} - r(n-r) - \frac{r(r-1)}{2} \\
&= \frac{(n-r)(n-r-1)}{2} \\
&= \dim \operatorname{Ext}_Q(V, V)^{(-\varphi)}
\end{aligned}$$

3.2.2. *Codimension of Symmetric Representations of A_3.* The next simplest symmetric quiver one can study is A_3. We fix a symmetric dimension vector α and a symmetric or skew-symmetric nondegenerate bilinear form $\langle\cdot, \cdot\rangle$ on K^α stisfying property (5). Then a representation $V \in \operatorname{Rep}_K(Q, \alpha)^\sigma$ has the form

$$V_1 \xrightarrow{V(a)} V_2 \xrightarrow{-V(a)^\star} V_1^*$$

We write V as a direct sum of its indecomposable modules:

$$V = E_{11}^{s_1} \oplus E_{22}^{s_2} \oplus E_{33}^{s_3} \oplus E_{12}^{c} \oplus E_{23}^{d} \oplus E_{13}^{e}$$

Since V is a symmetric quiver, it is not hard to check that we must have $s_1 = s_3$ and $c = d$ and s_3 and c must both be even if n is odd and $\langle\cdot, \cdot\rangle$ is symplectic or n is even and $\langle\cdot, \cdot\rangle$ is orthogonal. Now the symmetric quiver representation is uniquely defined by the following data: $\dim V_1 = m$, $\dim V_2 = n$,$\operatorname{rank} V(a) = r$ and $\operatorname{rank} V(a)^\star V(a) = s$. We relate these parameters to the expansion of V into indecomposables by:

$$e = s, \quad c = d = r - s, \quad s_1 = s_3 = m - r \quad \text{and} \quad s_2 = n - 2r + s$$

In order to carry out calculations, we need the dimensions of the vector spaces of Hom and Ext between the indecomposables representations.

$\dim \operatorname{Hom}(V_i, V_j)$

$V_i \backslash V_j$	E_{11}	E_{22}	E_{33}	E_{12}	E_{23}	E_{13}
E_{11}	1	0	0	0	0	0
E_{22}	0	1	0	1	0	0
E_{33}	0	0	1	0	1	1
E_{12}	1	0	0	1	0	0
E_{23}	0	1	0	1	1	1
E_{13}	1	0	0	1	0	1

$\dim \operatorname{Ext}(V_i, V_j)$

$V_i \backslash V_j$	E_{11}	E_{22}	E_{33}	E_{12}	E_{23}	E_{13}
E_{11}	0	1	0	0	1	0
E_{22}	0	0	1	0	0	0
E_{33}	0	0	0	0	0	0
E_{12}	0	0	1	0	1	0
E_{23}	0	0	0	0	0	0
E_{13}	0	0	0	0	0	0

With these preliminaries, we can state the following Lemma:

Lemma 3.6. *Let* V *be a symplectic or orthogonal representation of equioriented symmetric* A_3 *with* $\dim V_1 = m$, $\dim V_2 = n$, $\operatorname{rank} V(a) = r$ *and* $\operatorname{rank} V(a)^\star V(a) = s$. *Then*

$$\dim \operatorname{Ext}(V, V)^{(-\varphi)} = (m-r)(n-r) + \frac{(r-s)(r-s+\varepsilon)}{2} \tag{12}$$

Proof. Expressing V as a direct sum of indecomposable modules and using the above tables we calculate that $\dim \operatorname{Ext}(V, V) = 2(m-r)(n-r) + (r-s)^2$. Let φ be the linear involution on $\operatorname{Ext}(V, V)$. We see that $\varphi(\operatorname{Ext}(E_{12}, E_{23})) = \operatorname{Ext}(E_{12}, E_{23})$, $\varphi(\operatorname{Ext}(E_{11}, E_{22})) = \operatorname{Ext}(E_{22}, E_{33})$, and $\varphi(\operatorname{Ext}(E_{11}, E_{23})) = \operatorname{Ext}(E_{12}, E_{33})$. Thus we get:

$$\begin{aligned}\dim \operatorname{Ext}(V, V)^{(-\varphi)} &= \dim \operatorname{Ext}(E_{11}^{s_1}, E_{22}^{s_2}) + \dim \operatorname{Ext}(E_{11}^{s_1}, E_{23}^{c}) + \dim \Big(\operatorname{Ext}(E_{12}^c, E_{23}^c)^{(-\varphi)} \Big) \\ &= (m-r)(n-r) + \dim \Big(\operatorname{Ext}(E_{12}^c, E_{23}^c)^{(-\varphi)} \Big)\end{aligned}$$

Hence, the only nontrivial calculation we must carry out concerns the last term on the right.

We set $W = E_{12}^c \oplus E_{23}^c$ and restrict ourselves the the orthogonal (resp. symplectic) representation $(W, \langle \cdot, \cdot \rangle|_W)$. The dimension of $\big(\operatorname{Ext}(E_{12}^c, E_{23}^c)^{(-\varphi)} \big)$ is the same for any nondegenerate symmetric (resp. skew-symmetric) bilinear form $\langle \cdot, \cdot \rangle$. However, whether we are in the orthogonal case or the symplectic case, in order to carry out the calculations we need to utilize a specific bilinear form $\langle \cdot, \cdot \rangle$ on W_Σ that satisfies the conditions required for any symmetric quiver representation.

We choose the form on $V_\Sigma = V_1 \oplus V_2 \oplus V_3$ defined by $\langle v, w \rangle = {}^t v M w$ where

$$M = \begin{pmatrix} 0 & 0 & 0 & I_c \\ 0 & 0 & I_c & 0 \\ 0 & \varepsilon I_c & 0 & 0 \\ \varepsilon I_c & 0 & 0 & 0 \end{pmatrix}$$

The form $\langle \cdot, \cdot \rangle$ is symmetric if $\varepsilon = 1$ and skew-symmetric if $\varepsilon = -1$.

The adjoint operator $\star$ is now based on this matrix. A quick calculation shows that for a $2c \times c$ matrix $\begin{pmatrix} A \\ B \end{pmatrix} \in \operatorname{Hom}(K^c, K^{2c})$, we have

$$-\begin{pmatrix} A \\ B \end{pmatrix}^\star = \begin{pmatrix} -\varepsilon B^t & -A^t \end{pmatrix}$$

We note in particular that with the dimension vector $\alpha = (c, 2c, c)$ then the representation of $W = E_{12}^c \oplus E_{23}^c$ in $\mathrm{Rep}_K(Q, \alpha)$ can be given by

$$\left(\begin{pmatrix} I_c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -I_c \end{pmatrix} \right)$$

where I_c is the $c \times c$ identity matrix.

By the additivity of the Ext functor, we see that $\mathrm{Ext}(W, W) = \mathrm{Ext}(E_{12}^c, E_{23}^c)$. By definition,

$$\mathrm{Ext}(W, W) = \mathrm{coker}\Big(d : \mathrm{End}(K^c) \oplus \mathrm{End}(K^{2c}) \oplus \mathrm{End}(K^c) \longrightarrow \mathrm{Hom}(K^c, K^{2c}) \oplus \mathrm{Hom}(K^{2c}, K^c) \Big)$$

where,

$$d(M_1, M_2, M_3) = \left(\begin{pmatrix} I_c \\ 0 \end{pmatrix} M_1 - M_2 \begin{pmatrix} I_c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -I_c \end{pmatrix} M_2 - M_3 \begin{pmatrix} 0 & -I_c \end{pmatrix} \right)$$

We then see that $\mathrm{Im}(d) = \left(\begin{pmatrix} A \\ B \end{pmatrix}, \begin{pmatrix} B & C \end{pmatrix} \right)$ where A, B, C are any $c \times c$ matrices. Note in particular, that we can now read off $\dim \mathrm{Ext}(E_{12}^c, E_{23}^c) = c^2$. Thus

$$\mathrm{Ext}(E_{12}^c, E_{23}^c)^{(-\varphi)} \cong \mathrm{Hom}(K^c, K^{2c}) / \left\{ \begin{pmatrix} A \\ B \end{pmatrix} : \exists A, B, C \text{ with } \begin{pmatrix} -\varepsilon B^t & -A^t \end{pmatrix} = \begin{pmatrix} B & C \end{pmatrix} \right\}$$

Thus we deduce that

$$\dim \left(\mathrm{Ext}(E_{12}^c, E_{23}^c)^{(-\varphi)} \right) = \frac{c(c + \varepsilon)}{2}$$

We now return to the orthogonal (symplectic) quiver representation $(V, \langle \cdot, \cdot \rangle)$. Since $c = r - s$ in a general symmetric representation of A_3, we deduce that whether the form is orthogonal or symplectic, we have

$$\dim \mathrm{Ext}(V, V)^{(-\varphi)} = (m - r)(n - r) + \frac{1}{2}(r - s)(r - s + \varepsilon)$$

□

4. A Resolution of Singularities for Orbits of Representations of Symmetric A_n

In this section, we consider a symmetric quiver (Q, σ) a connected symmetric quiver of finite type, i.e. whose underlying graph Q is of type A_n.

Before we provide a construction for the desingularization of orbit closures, we need a degeneration characterization, that is to say a characterization of when one orbit is a subset of the closure of another.

Lemma 4.1. *Let A and B be two symmetric quiver representations. Then as orthogonal (resp. symplectic) orbits,*

$$O_B \subseteq \overline{O_A} \Longleftrightarrow \dim \mathrm{Hom}_Q(A, X) \leq \dim \mathrm{Hom}_Q(B, X)$$

for all symmetric representations X.

Proof. Since we discuss orbits of representations in different spaces, we set $G = GL_Q(\alpha)$ and denote the orbit of V in $\mathrm{Rep}_K(Q, \alpha)$ as $G \cdot V$ and the orbit of V under G^σ in $\mathrm{Rep}_K(Q, \alpha)^\sigma$ as $G^\sigma \cdot V$.

In the articles [5, 6], Bongartz constructs various partial orders on quiver representations in $\mathrm{Rep}_K(Q, \alpha)$. Per those articles, for Dynkin quivers, $B \subseteq \overline{G \cdot A}$ if and only if $\dim \mathrm{Hom}_Q(A, X) \leq \dim \mathrm{Hom}_Q(B, X)$ for all modules X.

($\Rightarrow$) This direction is easy since $B \subseteq \overline{G^\sigma \cdot A} \subseteq \overline{G \cdot A}$ and thus

$$\dim \mathrm{Hom}_Q(A, X) \leq \dim \mathrm{Hom}_Q(B, X)$$

($\Leftarrow$) Suppose $\dim \mathrm{Hom}^s_Q(A,X) \leq \dim \mathrm{Hom}^s_Q(B,X)$. By [6], we have $G\cdot B \subseteq \overline{G\cdot A}$ where G is the usual group of the underlying quiver. By Theorem 2.7, if $V \in \mathrm{Rep}_K(Q,\alpha)^\sigma$, then $G^\sigma \cdot V = G\cdot V \cap \mathrm{Rep}_K(Q,\alpha)^\sigma$. Thus $G^\sigma \cdot B = G\cdot B \cap \mathrm{Rep}_K(Q,\alpha)^\sigma \subseteq \overline{G\cdot A} \cap \mathrm{Rep}_K(Q,\alpha)^\sigma$.

However, in any topological space X, if S and T are any sets but with T closed, then $\overline{S\cap T} = \overline{S} \cap T$.

In our situation, $\mathrm{Rep}_K(Q,\alpha)^\sigma$ is a closed subvariety of $\mathrm{Rep}_K(Q,\alpha)$ since it is a linear subspace. Consequently, $\overline{G\cdot A} \cap \mathrm{Rep}_K(Q,\alpha)^\sigma = \overline{G\cdot A \cap \mathrm{Rep}_K(Q,\alpha)^\sigma} = \overline{G^\sigma \cdot A}$. The result follows. $\square$

In his paper [12], Reineke provides a general method to construct desingularizations of orbit closures for all representations of standard quivers of finite type. His proof hinges on the directedness of the category KQ-$\mathfrak{Mod}$ established by Ringel in [13], a property that no longer holds in KQ-$\mathfrak{Mod}_o$ or KQ-$\mathfrak{Mod}_s$. However, making a few judicious choices and some slight modifications to Reineke's construction, we can still obtain a desingularization of symmetric quivers of finite type.

We remind the reader of key elements in Reineke's construction. Consider a standard quiver Q.

Given sequences $\mathbf{i} = (i_1, \ldots, i_\nu) \in Q_0^\nu$ and $\mathbf{a} = (a_1, \ldots, a_\nu) \in \mathbb{N}^\nu$, we call the pair $(\mathbf{i}, \mathbf{a})$ a **monomial** for the quiver Q. The **weight** of the monomial $(\mathbf{i},\mathbf{a})$ is the dimension vector $\sum_k a_k i_k \in \mathbb{N}^{Q_0}$. We say that a monomial is **pure of weight** i if the support of the weight of the monomial is the single vertex $i \in Q_0$.

Let $(\mathbf{i},\mathbf{a})$ be a monomial of weight α. We define the flag variety $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ as the set of Q_0-graded flags

$$F: \qquad K^\alpha = F^0 \supset F^1 \supset \ldots \supset F^\nu = 0 \tag{13}$$

such that F^{k-1}/F^k is pure of weight i_k and dimension a_k, i.e. is a subspace of $K^\alpha_{i_k}$ of dimension a_k. The group $GL_Q(\alpha)$ acts transitively on the set of flags $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ and hence if we are given a fixed flag $F_0 \in \mathcal{F}_{\mathbf{i},\mathbf{a}}$ and denote its stabilizer as $P_{\mathbf{i},\mathbf{a}}$, then $\mathcal{F}_{\mathbf{i},\mathbf{a}} \cong GL_Q(\alpha)/P_{\mathbf{i},\mathbf{a}}$.

Definition 4.2. If Q is a standard quiver, a partition $\mathcal{I} = (\mathcal{I}_1 \ldots \mathcal{I}_s)$ of the positive roots of the quiver Q is called *directed* if

1. $\mathrm{Ext}_Q(X_\alpha, X_\beta) = 0$ for all $\alpha, \beta \in \mathcal{I}_t$, for $t = 1, \ldots s$;
2. $\mathrm{Hom}_Q(X_\beta, X_\alpha) = 0 = \mathrm{Ext}_Q(X_\alpha, X_\beta)$ for all $\alpha \in \mathcal{I}_t$, $\beta \in \mathcal{I}_u$ with $t < u$.

Remark 4.1. In [13], Ringel shows that if Q is of finite type, the category KQ-$\mathfrak{Mod}$ is representation-directed ensuring that directed partitions always exist.

Given a representation $V \in \mathrm{Rep}_K(Q,\alpha)$ and a directed partition $\mathcal{I} = (\mathcal{I}_1 \ldots \mathcal{I}_s)$ of the roots of Q, we construct a particular monomial $(\mathbf{i},\ \mathbf{a}(V))$ which allows us to obtain a desingularization of the orbit closure of $G\cdot V$ in $\mathrm{Rep}_K(Q,\alpha)$. Since Q possesses no oriented cycles, we establish a total order $<$ on the vertices Q_0 such that the existence of an arrow $i \longrightarrow j$ implies $i < j$. For $1 \leq t \leq s$ we define ω_t as the sequence in Q_0 obtained by taking the elements of

$$\{i \in Q_0 : \alpha_i \neq 0 \text{ for some } \alpha \in \mathcal{I}_t\}$$

in increasing order. Then we define $\mathbf{i}$ as the concatenation $(\omega_1 \ldots \omega_s)$.

For any $V \in KQ$-$\mathfrak{Mod}$, we define $V_{(t)}$ as the direct sum of all summands U of V which are isomorphic to some X_α where $\alpha \in \mathcal{I}_t$. Then, writing ω_t as $(i_1, \ldots, i_u)$ we define $\mathbf{a}_t(V)$ as the sequence $(\dim_{i_1} V_{(t)}, \ldots, \dim_{i_u} V_{(t)})$. Finally, we define the sequence $\mathbf{a}(V)$ as the concatenation $(\mathbf{a}_1(V), \ldots, \mathbf{a}_s(V))$ and we call the pair the pair $(\mathbf{i}, \mathbf{a}(V))$ the monomial for V with respect to the partition $\mathcal{I}$. Clearly

$$V = V_{(1)} \oplus \cdots \oplus V_{(s)} \qquad \text{and} \qquad \mathbf{dim}\, V = \sum_{k=1}^{|\mathbf{i}|} a_k i_k \in \mathbb{N}Q_0$$

We will call the monomial $(\mathbf{i}, \mathbf{a}(V))$ constructed as above, the *monomial subordinate to the partition $\mathcal{I}$ and the representation V.*

In order to construct a desingularization for orbit closures of symmetric quiver representations, we follow Reineke's setup but employ a specific partition of roots that is symmetric under the action of our involution and modify the proof of his Theorem 2.2 in [12].

First, we need to establish the notion of symmetric monomials. For any finite sequence u_k of length $|u|$, define $\rho(u)$ as the reverse sequence that has $\rho(u)_k = u_{|u|+1-k}$ for $1 \leq k \leq |u|$. Then we call the monomial $(\mathbf{i},\mathbf{a})$ symmetric if $\rho(\mathbf{i})_k = \sigma(\mathbf{i}_k)$ and $\rho(\mathbf{a}) = \mathbf{a}$.

We also need to utilize a partition of roots for Q that is directed for the underlying quiver and that is also symmetric with respect to our involution σ on $\mathrm{Rep}_K(Q,\alpha)$.

Definition 4.3. Let $\mathcal{I} = \{\mathcal{I}_1, \dots, \mathcal{I}_s\}$ be a directed partition of positive roots of a symetric quiver (Q,σ). We call a directed partition *symmetric* if for all $t \in \{1,\dots,s\}$,

$$\rho(\mathcal{I})_t = \left\{(\alpha_{\sigma(i)})_{i\in Q_0} \ : \ \alpha \in \mathcal{I}_t\right\}$$

In otherwords, $\rho(\mathcal{I})_t = \sigma(\mathcal{I}_t)$.

For any definition to carry weight, one must know that such an object exists. Hence, we prove that symmetric directed partitions exist for symmetric quivers of finite type by illustrating one partition with all the prescribed properties.

Definition 4.4. Let Q be a quiver of finite type. Consider the Coxeter functors τ^+ and τ^- on KQ-$\mathfrak{Mod}$ as defined in [4] or [2]. Define a τ^--chain (resp. τ^+-chain) as a sequence of isomorphism classes of indecomposable modules $(M_i)_{1\leq i\leq l}$ such that $M_{i+1} = \tau^-(M_i)$ (resp. $M_{i+1} = \tau^+(M_i)$). Moreover, we define a *maximal* τ^--chain (resp. τ^+-chain) as a τ^--chain (resp. τ^+-chain) of maximal length in the set of all chains of Coxeter functors.

Remark 4.2. Note that for any Dynkin quiver, regardless of orientation, a maximal τ^+-chain begins at an injective simple module and ends at a projective simple module, and vice versa for τ^--chains.

Definition 4.5. Let L be an indecomposable quiver module which is not injective. We denote by $J^+(L)$ the module in KQ-$\mathfrak{Mod}$ such that a representative of the non-trivial element in $\mathrm{Ext}(\tau^+(L), L)$ in given by:

$$0 \longrightarrow \tau^+(L) \longrightarrow J^+(L) \longrightarrow L \longrightarrow 0$$

The modules $J^+(L)$ are usually not indecomposable but we shall use them in the following to construct a symmetric partition.

Lemma 4.6. *Let (Q,σ) be a symmetric quiver and let $(L_{s'}, \dots, L_2, L_1)$ be a maximal τ^+-chain of classes of indecomposable representations of the underlying quiver Q^o. Suppose we impose upon a directed partition of roots $\mathcal{I}_*$ the following two conditions:*

- $\mathbf{dim}\, L_t \in \mathcal{I}_{2t-1}$ *for* $1 \leq t \leq s'$;
- *if* L' *is an indecomposable summand of* $J^+(L)$ *where* $\mathbf{dim}\, L \in \mathcal{I}_t$ *then* $\mathbf{dim}\, L' \in \mathcal{I}_{t-1}$.

Then such a directed partition $\mathcal{I}_*$ exists and is a symmetric directed partition of length $s = |\mathcal{I}_*| = 2s' - 1$.

Proof. Since $\tau^+(L_1) = 0$ and since there exists no module $M \in KQ$-$\mathfrak{Mod}$ such that $\tau^+(M) = L_{s'}$, then if the two conditions hold, they together impose that $|\mathcal{I}_*| = 2s' - 1 := s$.

Since KQ-$\mathfrak{Mod}$ is representation-directed, there exist no cycles in the Auslander-Reiten quiver of indecomposable modules of the path algebra KQ. Therefore, by properties of the AR-quiver (see [3]), if M and N are indecomposable modules in KQ-$\mathfrak{Mod}$, then all paths from $[M]$ to $[N]$ have the same length.

We remark that by Proposition 3.4 in [8] states that the involution functor and the Coxeter functors are related by

$$\sigma\tau^{\pm} = \tau^{\mp}\sigma \tag{14}$$

Consequently, the AR-quiver itself is a symmetric quiver where for all indecomposable modules M we have $\sigma([M]) = [M^*]$ (the dual quiver after relabeling $i \to \sigma(i)$) and for all irreducible morphism $\alpha : M \to N$, $\sigma(\alpha) = \alpha^*$.

By Gabriel's Theorem, since Q is finite type, there is a 1-1 correspondence between positive roots and indecomposable modules so the vertices of the AR-quiver are parametrized by positive roots. A partition $\mathcal{I}_*$ of positive roots satisfying the conditions of the lemma can be constructed as follows. $\mathcal{I}_1$ (resp. $\mathcal{I}_s$) consists of dimension vectors of final (resp. initial) terms in maximal τ^+-chains. (Note that final terms of τ^+-chains must be projective.) Then impose the condition that if there exists an irreducible morphsim $f : M \to N$ for $M, N \in \operatorname{Ind} KQ$ and $\mathbf{dim}\, M \in \mathcal{I}_t$, then $\mathbf{dim}\, N \in \mathcal{I}_{t+1}$. By Theorem VII, 2.1 in [3] the AR-quiver is connected and since all paths between pairs of modules have the same length, this partition $\mathcal{I}_*$ is well-defined.

For every indecomposable module L and for every indecomposable summand N of $J^+(L)$, we know that

$$\tau^+(L) \to N \qquad \text{and} \qquad N \to L$$

are irreducible morphisms. Consequently, $\mathcal{I}_*$ satisfies the desired conditions of the lemma.

We must check that the partition $\mathcal{I}_*$ is directed. Consider indecomposable modules M and N such that $\mathbf{dim}\, M \in \mathcal{I}_t$ and $\mathbf{dim}\, N \in \mathcal{I}_u$ with $t \leq u$. Then there does not exist a path

$$p : N = A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n = M$$

in the AR-quiver. Consequently, $\operatorname{Ext}_Q(M, N) = 0$. By the same reasoning, we also obtain that $\operatorname{Hom}_Q(N, M) = 0$. Therefore, the partition we constructed is directed.

Finally, our partition is symmetric by construction, that is to say that for $1 \leq t \leq s$, the roots in $\mathcal{I}_{s+1-t}$ (reversing the sequence) are precisely the roots of elements in $\sigma(\mathcal{I}_t)$. ☐

Remark 4.3.
(1) Note that these partitions constructed in Lemma 4.6 correspond to vertical columns in the Auslander-Reiten quiver when displayed in the standard way. For the reader who is acquainted with the Auslander-Reiten quiver, this remark provides a very simple way to construct symmetric directed partitions in a specific example.
(2) We remark that we may also characterize this partition as the minimal partition in which for all indices t with $1 \leq t \leq |\mathcal{I}_*|$ we have $\operatorname{Hom}_Q(X_\alpha, X_\beta) = 0 = \operatorname{Ext}_Q(X_\alpha, X_\beta)$ for all $\alpha \neq \beta$ in $\mathcal{I}_t$.
(3) Finally, the partition constructed in Lemma 4.6 is not always the only symmetric directed partition that exists. In fact, one may profit from a different partition when studying orbits of a symmetric A_3 quiver in great detail. (The author will make use of this in a future article that discusses orthogonal and symplectic orthogonal ideals.)

Lemma 4.7. *Let $\mathcal{I}$ be a symmetric directed partition of the positive roots of a symmetric quiver (Q, σ). Then for any orthogonal or symplectic representation $(V, \langle \cdot, \cdot \rangle)$ of Q, the monomial $(\mathbf{i}, \mathbf{a}(V))$ subordinate to $\mathcal{I}$ and V is a symmetric monomial.*

Proof. Following the notation of the construction $(\mathbf{i}, \mathbf{a}(V))$, $\mathbf{i} = (\omega_1 \omega_2 \ldots \omega_s)$. Thus, reversing the sequence we get $\rho(\mathbf{i}) = (\rho(\omega_s)\rho(\omega_{s-1}) \ldots \rho(\omega_1))$. For any $1 \leq t \leq s$, $\rho(\omega_{s+1-t})$ consists of $i \in Q_0$ such that $\alpha_i \neq 0$ for $\alpha \in \mathcal{I}_{s+1-t}$ written in reverse order. $\sigma(\rho(\omega_t))$ consists of $\sigma(i)$ such that $\alpha_i \neq 0$ for $\alpha \in \mathcal{I}_{s+1-t}$ that is of i such that $\alpha_i \neq 0$ for $\alpha \in \mathcal{I}_t$ written in increasing order. Hence $\sigma(\rho(\mathbf{i})) = \mathbf{i}$ and hence $\mathbf{i}$ is symmetric.

The same work applies to show that if $(V, \langle \cdot, \cdot \rangle)$ is an orthogonal or symplectic representation, then $\mathbf{a}(V)$ is symmetric. ☐

Reineke's flag variety $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ defined above is not quite the space we need in our situation. We define the variety of symmetric flags, $\mathcal{F}^s_{\mathbf{i},\mathbf{a}}$ as follows. If $(\mathbf{i}, \mathbf{a})$ is a symmetric monomial of length ν, then

$\mathcal{F}^s_{\mathbf{i},\mathbf{a}}$ consists of the flags in $\mathcal{F}_{\mathbf{i},\mathbf{a}}$

$$K^\alpha = F^0 \supset F^1 \supset \ldots \supset F^\nu = 0 \tag{15}$$

that also satisfy $F^{\nu-j} = (F^j)^\perp$ where perpendicularity of course depends on the form $\langle \cdot, \cdot \rangle$ on V_Σ. We notice that such symmetric flags are in a one-to-one correspondance with the set of isotropic flags in V_Σ.

Define $X^s_{\mathbf{i},\mathbf{a}}$ as pairs of symmetric representations and symmetric flags that are compatible. That is to say,

$$X^s_{\mathbf{i},\mathbf{a}} = \{(R,F) \in \mathrm{Rep}_K(Q,\alpha)^\sigma \times \mathcal{F}^s_{\mathbf{i},\mathbf{a}} : R(F^i) \subset F^{i+1}\} \tag{16}$$

We fix one particular symmetric flag, say $F_0 \in \mathcal{F}^s_{\mathbf{i},\mathbf{a}}$ and let $P_{\mathbf{i},\mathbf{a}}$ be its stabilizer in G^σ. We call $Y^s_{\mathbf{i},\mathbf{a}}$ the subspace of representations in $\mathrm{Rep}_K(Q,\alpha)^\sigma$ that are compatible with F_0.

With these notations, we remark that the elements of G^σ preserve the form $\langle \cdot, \cdot \rangle$ and acts transitively on the variety of symmetric flags $\mathcal{F}^s_{\mathbf{i},\mathbf{a}}$ and thus we can identify $G^\sigma / P_{\mathbf{i},\mathbf{a}}$ with $\mathcal{F}^s_{\mathbf{i},\mathbf{a}}$. Furthermore, we identify $X^s_{i,a}$ with

$$G^\sigma \times^{P_{\mathbf{i},\mathbf{a}}} Y^s_{\mathbf{i},\mathbf{a}} = G^\sigma \times Y^s_{\mathbf{i},\mathbf{a}} \Big/ \Big((g,V) \sim (gh, hV) \text{ with } h \in P_{\mathbf{i},\mathbf{a}} \Big)$$

via:

$$\begin{aligned} G^\sigma \times^{P_{\mathbf{i},\mathbf{a}}} Y^s_{\mathbf{i},\mathbf{a}} &\longrightarrow X^s_{\mathbf{i},\mathbf{a}} \\ \overline{(g,M)} &\longmapsto (gMg^{-1}, gF_0) \end{aligned}$$

This identification presents $X^s_{\mathbf{i},\mathbf{a}}$ as a vector bundle over the homogeneous space of symmetric flags $\mathcal{F}^s_{\mathbf{i},\mathbf{a}} = G^\sigma / P_{\mathbf{i},\mathbf{a}}$ so that the map

$$\pi_{\mathbf{i},\mathbf{a}} : X^s_{\mathbf{i},\mathbf{a}} \longrightarrow \mathrm{Rep}_K(Q,\alpha)^\sigma$$

that projects onto the first factor in the definition of $X^s_{\mathbf{i},\mathbf{a}}$ is a projective morphism. We can now state the main result of this section, basing its proof in large part on the proof of Reineke's theorem 2.2 in [12].

Theorem 4.8. *Let $\mathcal{I}$ be a symmetric directed partition. Let V be a symplectic or orthogonal representation of Q and let $(\mathbf{i},\mathbf{a})$ be the monomial subordinate to $\mathcal{I}$ and V.*

Then the image of $X^s_{\mathbf{i},\mathbf{a}}$ under $\pi_{\mathbf{i},\mathbf{a}}$ is the orbit closure $\overline{O}_V = \overline{G^\sigma \cdot V}$ and $\pi_{\mathbf{i},\mathbf{a}}$ provides a desingularization of $\overline{O}_V$.

Proof. First, we remark that as in the constructions of [1] and [12], since $X^s_{\mathbf{i},\mathbf{a}}$ is a vector bundle over a homogeneous space it is a smooth scheme. Furthermore, $X^s_{\mathbf{i},\mathbf{a}}$ is a closed subscheme of $G^\sigma/P_{\mathbf{i},\mathbf{a}} \times Y^s_{\mathbf{i},\mathbf{a}}$ and so $\pi_{\mathbf{i},\mathbf{a}}$ is projective and hence proper.

We now prove $\pi_{\mathbf{i},\mathbf{a}}(X^s_{\mathbf{i},\mathbf{a}}) = \overline{O}_V$. Now consider a pair $(M, F = \{F_j\}) \in X^s_{\mathbf{i},\mathbf{a}}$. Since the two elements in the pair are compatible, we have $M(F_k) \subseteq F_{k+1}$. The purpose in using a partition is that what may use the method of Reineke's proof to deduce the key step that this compatibility implies $M \subseteq \overline{G \cdot V}$. Then, by Lemma 4.1, we deduce that $\pi_{\mathbf{i},\mathbf{a}}(X^s_{\mathbf{i},\mathbf{a}}) \subseteq \overline{G^\sigma \cdot V} = \overline{O}_V$.

However, by construction $\pi^{-1}_{\mathbf{i},\mathbf{a}}(V) \in X^s_{\mathbf{i},\mathbf{a}}$ and the space $X^s_{i,a}$ is G^σ-stable. Hence $\pi_{\mathbf{i},\mathbf{a}}(X_{\mathbf{i},\mathbf{a}}) = \overline{O}_V$.

Finally, we claim that $\pi_{\mathbf{i},\mathbf{a}} : \pi^{-1}_{\mathbf{i},\mathbf{a}}(O_V) \longrightarrow O_V$ is an isomorphism. However, we only need to show that $\pi^{-1}_{\mathbf{i},\mathbf{a}}(\pi_{\mathbf{i},\mathbf{a}}(M)) = M$ where $M = M_0 \supset M_1 \supset \ldots \supset M_\nu = 0$ is a filtration of type $(\mathbf{i}, \mathbf{a}(V))$. We can coarsen the filtration to $M = \tilde{M}_0 \supset \tilde{M}_1 \supset \ldots \supset \tilde{M}_s = 0$ where $\tilde{M}_{k-1}/\tilde{M}_k$ is of dimension vector

$$\mathbf{dim}\, M_{(t)} = \sum_{j \in \mathcal{I}_t} a_j i_j \in \mathbb{N}Q_0 \tag{17}$$

By Lemma 2.3 in [12] and the Hom-vanishing and Ext-vanishing properties coming from our choice of symmetric directed partition, we see that $\tilde{M_t} \cong M_{(t+1)} \oplus \cdots \oplus M_{(s)}$ for all t and that this filtration is unique.

In the situation of symmetric representations of A_n the only remaining difference from standard representations comes from the fact that for $t \neq \frac{s+1}{2}$, $M_{(t)}$ is not a symmetric subrepresentation. However, we obtain unique orthogonal or symplectic subrepresentations by taking instead $M'_{(t)} = M_{(t)} \oplus M_{(s+1-t)}$ for $1 \leq t < \frac{s+1}{2}$ and simply $M'_{(t)} = M_{(t)}$ if $t = \frac{s+1}{2}$.

This shows that $\pi_{\mathbf{i},\mathbf{a}} : \pi^{-1}_{\mathbf{i},\mathbf{a}}(O_V) \longrightarrow O_V$ is an isomorphism and hence concludes the proof of the theorem. ☐

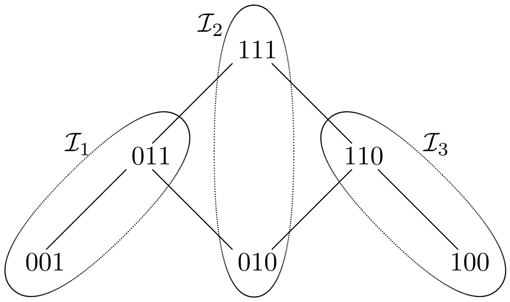
5. Explicit Calculations for Symmetric Equioriented A_n

5.1. **Desingularization of orbits of a symmetric A_3 quiver.** We return to Example 2.17 and describe the desingularization for orbits of symmetric A_3 quivers. Number the vertices of the underlying quiver as $A_3 : 1 \longrightarrow 2 \longrightarrow 3$.

To use Theorem 4.8, we first must have a symmetric directed partition of the roots of A_3. In practice, it is easy to find a such a partition by looking at the Auslander-Reiten quiver. Because of (14), the AR-quiver of a symmetric quiver (Q, σ) will possess a natural axis of symmetry. We must select a directed partition of the roots that is symmetric with respect to this axis.

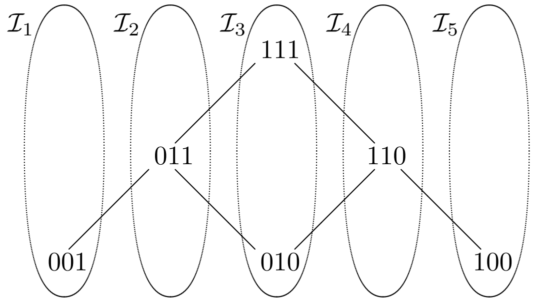
The following diagram illustrates the coarsest symmetric directed partition on A_3:

(18)



The above symmetric directed partition is not unique for we also could have chosen the following:

(19)



In order to construct a minimal desingularization, we will use the symmetric directed partition $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ listed in the first diagram. This partition $\mathcal{I}$ gives us the sequence of vertices

$$\mathbf{i} = (2, 3, 1, 2, 3, 1, 2)$$

Now let $(V, \langle\cdot,\cdot\rangle)$ be an orthogonal (resp. symplectic) representation of symmetric A_3 with (symmetric) dimension vector $\mathbf{dim}V = (r_{11}, r_{22}, r_{11})$. We also call

$$r_{ij} = \operatorname{rank} V(i \to j)$$

for $1 \le i < j \le 3$ and by symmetry $r_{23} = r_{12}$. Using a standard result from [1], we find that the corresponding sequence $\mathbf{a}(V)$ is

$$\mathbf{a}(V) = (r_{12} - r_{13}, r_{11} - r_{13}, r_{13}, r_{22} - 2r_{12} + 2r_{13}, r_{13}, r_{11} - r_{13}, r_{11} - r_{12})$$

As a corollary to Theorem 4.8, we can exhibit a desingularization of the closure of the orbit of V in $\mathrm{Rep}_K(Q, \alpha)^\sigma$.

Corollary 5.1. *Let V be an orthogonal (resp. symplectic) representation as above. Let $\mathcal{F}$ be the homogeneous space $\mathcal{F} = \mathbb{G}(r_{11} - r_{13}, V_1) \times \mathbb{IG}(r_{12} - r_{13}, V_2)$ - a product of a Grassmannian and an isotropic Grassmannian. Then the desingularization of the orbit closure $\bar{O}_V$ in $\mathrm{Rep}_K(Q, \alpha)^\sigma$ is*

$$X = \{((R, S), \phi) \in \mathcal{F} \times \mathrm{Hom}(V_1, V_2) \,:\, \phi(V_1) \subseteq \check{S} \text{ and } \phi(R) \subseteq S\}$$

where $\check{S} = \{v \in V_2 \,:\, \langle w, v\rangle = 0$ for all $w \in S\}$.

Proof. For U any subspace of V_Σ, define $\check{U} = \{v \in V_\Sigma \,:\, \langle w, v\rangle = 0$ for all $w \in U\}$. By the properties of $\langle\cdot,\cdot\rangle$, if U is a subspace of V_i then $\check{U}$ is a subspace of $V_{\sigma(i)}$. Furthmore, it's easy to check that

$$\phi(U_1) \subset U_2 \iff \phi^\star(\check{U}_2) \subset \check{U}_1$$

From Example 2.17 we see that $\mathrm{Rep}_K(Q, \alpha)^\sigma = \mathrm{Hom}(V_1, V_2)$. Let $\mathcal{F}_{\mathbf{i},\mathbf{a}(V)}$ be the flag variety as described in (13). We now interpret the definition of $X^s_{\mathbf{i},\mathbf{a}(V)}$ in (16). Let F be a symmetric flag in $\mathcal{F}_{\mathbf{i},\mathbf{a}(V)}$. Since $(\phi, \phi^\star)(F^0) \subset F^1$ and F^0/F^1 is of pure weight 2 and dimension $r_{12} - r_{13}$, we deduce that $\phi(V_1) \subset \check{S}$ where $\check{S}$ is a subspace of V_2 of dimension $r_{22} - r_{12} + r_{13}$. We also deduce that $\phi^\star(S) = 0$ which implies that S is an isotropic space of V_2.

Since $(\phi, \phi^\star)(F^1) \subset F^2$ and F^1/F^2 is of pure weight 3 and dimension $r_{11} - r_{13}$, we deduce that $\phi^\star(\check{S}) \subset \check{R}$ where $\check{R}$ is a subspace of V_3 of dimension r_{13}. Consequently, $\phi(R) \subset S$ where R is a subspace of V_1 of dimension $r_{11} - r_{13}$. It is now only a matter of exhaustion to show that the condition $(\phi, \phi^\star)(F^i) \subset F^{i+1}$ adds no new restrictions on subspaces. ☐

We leave as an exercise to the reader to verify that

$$\operatorname{codim} \bar{O}_V = \dim \mathrm{Rep}_K(Q, \alpha)^\sigma - \dim X = \dim \mathrm{Ext}(V, V)^{(-\varphi)}$$

using the formula calculated in Lemma 3.6.

5.2. **Properties of the Symmetric Directed Partition for Equioriented A_n.** Let us briefly consider the more general case of equioriented A_n so that representationshave the form

$$V : \quad V_1 \xrightarrow{V(a_1)} V_2 \xrightarrow{V(a_2)} \cdots \xrightarrow{V(a_{n-1})} V_n$$

Similar to the presentation in [1], for all $(i, j) \in T_n = \{(i, j) \,:\, 1 \le i \le j \le n\}$ define

$$r^V_{ij} = \operatorname{rank} V(i \to j) \qquad \text{for all } (i, j) \in T_n$$

In order to simplify formulas, we also set $r^V_{ij} = 0$ for all $(i, j) \notin T_n$.

Let $(V, \langle\cdot,\cdot\rangle)$ be a symplectic or orthogonal representation of symmetric equioriented A_n quiver. As a KQ-module, V decomposes into the direct sum

$$V \cong \bigoplus_{(i,j)\in T_n} E_{ij}^{\lambda_{ij}}$$

with

$$\lambda_{ij} = r_{ij}^V - r_{i-1,j}^V - r_{i,j+1}^V + r_{i-1,j+1}^V \qquad \text{for all } (i,j) \in T_n$$

For equioriented A_n in general, the partition exhibited in (18) is no longer applicable but the partition in (19) always is. We use this second partition where the sets of roots correspond to vertical columns in the AR-quiver. Explicitly, this partition is $\mathcal{I}_* = (\mathcal{I}_1, \ldots, \mathcal{I}_s)$ where $s = 2n-1$ and

$$\mathcal{I}_k = \{\alpha : X_\alpha \cong E_{ij} \in \mathrm{Rep}_K(Q,\alpha) \text{ where } k = 2n+1-i-j\}$$

With this partition and an orthogonal or symplectic representation $V \in \mathrm{Rep}_K(Q,\alpha)^\sigma$ we create a monomial $(\mathbf{i}, \mathbf{a}(V))$. Recalling that $\mathbf{i} = (\omega_1 \ldots \omega_{2n-1})$, we can use a simple induction to show that $\omega_t = (\max\{1, n+1-t\}, \ldots, \min\{2n-t, n\})$ and hence that $|\omega_t| = \min\{t, 2n-t\}$.

From these two observations we deduce two more facts. First of all, each vertex $k \in Q_0$ appears exactly n times in $\mathbf{i}$. Secondly, and as a consequence of the first observation, we have

$$\sum_{t=1}^{2n-1} |\omega_t| = n^2 \tag{20}$$

so $\mathbf{i}$ and $\mathbf{a}$ have length n^2.

The calculations for $a_t(V)$ given us

$$(a_t(V))_u = \sum_{\substack{i+j=2n+1-t;\\ i\le u\le j}} \lambda_{ij}^V \qquad \text{for } 1 \le u \le |\omega_t| \tag{21}$$

Furthermore, for each vertex $k \in Q_0$ we can project the flag variety $\mathcal{F}_{\mathbf{i},\mathbf{a}}^s$ onto a certain flag variety of subspaces of V_k. Since each vertex k appears in $\mathbf{i}$ n times, then on each vertex every flag in $\mathcal{F}_{\mathbf{i},\mathbf{a}}^s$ restricts to a flag of the form $V_k = F^0 \supset F^1 \supset \cdots \supset F^n = 0$ where

$$\dim(F^k/F^{k+1}) = \sum_{\substack{i+j=n+l-k;\\ i\le l\le j}} \lambda_{ij}^V \qquad \text{where } 0 \le l \le n-1 \tag{22}$$

Furthermore, if $n = 2n'-1$ is odd, then $\mathcal{F}_{\mathbf{i},\mathbf{a}}^s$ projects onto a symmetric flag variety of subspaces in $V_{n'}$ which in turn can be identified with the isotropic flag $0 = F^0 \subset F^1 \subset \cdots \subset F^{n'}$ where

$$\dim F_l = \sum_{\substack{n'+1\le i+j\le n'+l;\\ i\le n'\le j}} \lambda_{ij} \tag{23}$$

5.3. **A Numerical Codimension Formula.** As an immediate result of theorem 4.8, we can deduce the dimension and hence codimension of the of the orbit closure of an equioriented symmetric A_n-representation V.

We recall for comparison the codimension of the orbit of a standard A_n representation with rank conditions r_{ij}, where $1 \le i \le j \le n$. As noted in [7] if one considers representations of a standard equioriented A_n quiver, the codimension of the orbit in $\mathrm{Rep}_K(Q,\alpha)$ is:

$$\sum_{i<j} (r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij}) \tag{24}$$

For symmetric representations we obtain instead:

Corollary 5.2. *Setting* $\varepsilon' = (-1)^{n-1}\varepsilon$ *and* $\bar{T}_n = \{(i,j) : 1 \le i \le j \le n \text{ and } i+j < n+1\}$*, the codimension of* V*, a representation of a symmetric quiver* Q*, is:*

$$\operatorname{codim}(\overline{O}_V) = \sum_{\substack{(i,j)\in\bar{T}_n \\ i+j\neq n+1}} (r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij}) + \sum_{1\le i\le \frac{n+1}{2}} \frac{1}{2}(r_{i,\sigma(i)-1} - r_{i,\sigma(i)})(r_{i,\sigma(i)-1} - r_{i,\sigma(i)} + \varepsilon') \tag{25}$$

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