

# RESOLUTIONS OF ORTHOGONAL AND SYMPLECTIC ANALOGUES OF DETERMINANTAL IDEALS

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ABSTRACT. We consider orthogonal and symplectic analogues of determinantal varieties  $\bar{O}_{r_1, r_2}$ . Such varieties simultaneously generalize usual determinantal varieties and rank varieties of symmetric or anti-symmetric matrices. We find (non-minimal) resolutions of the coordinate rings of the varieties  $\bar{O}_{r_1, r_2}$ . We determine that “nearly all” such varieties are Cohen-Macaulay and for those that are Cohen-Macaulay we calculate the type. Furthermore, we provide a simple characterization for which varieties  $\bar{O}_{r_1, r_2}$  are Gorenstein. As an application, we present a class of ideals in  $k[\text{Hom}(E, F)]$  that are Gorenstein of codimension 4.

## 1. INTRODUCTION

In [5], Kac classified all the representations of connected reductive groups with finitely many orbits. In his article, Kac organizes representations with finitely many orbits into three types and lists them explicitly in his Tables II, III and IV. Among the representations on Table II, one finds only four families of doubly-infinite families. Setting  $k$  an algebraically closed field of characteristic 0, these four families are

- a)  $G = SL(E) \times SL(F) \times k^*$  acting on  $V = E \otimes F$  where  $E = k^e$  and  $F = k^f$ ;
- b)  $G = GL(E) \times SO(F)$  acting on  $V = E \otimes F$  where  $E = k^e$  and  $F = k^f$  with  $F$  an orthogonal space and  $f$  odd;
- c)  $G = GL(E) \times Sp(F)$  acting on  $V = E \otimes F$  where  $E = k^e$  and  $F = k^f$  with  $F$  a symplectic space (and  $f$  even);
- d)  $G = GL(E) \times SO(F)$  acting on  $V = E \otimes F$  where  $E = k^m$  and  $F = k^f$  with  $F$  an orthogonal space and  $f$  even.

Closures of orbits in family (1) correspond to determinantal varieties about which much is known. The result that is most relevant to this paper is Lascoux’s minimal free resolution of the determinantal varieties established in [8]. Given two  $e, f$  two integers, one sets  $X = \text{Hom}(k^e, k^f) = \text{Hom}(E, F)$  and one considers the subvariety  $Y_r$  of all matrices of rank at most  $r$ . Lascoux defines a minimal free resolution of  $k[Y_r]$  over  $A = k[X]$  where the  $i$ -th term in the resolution is given by:

$$(1) \quad F_i = \bigoplus_{s \geq 0} \bigoplus_{\lambda \in P(r, s): |\lambda| - rs = i} K_{\lambda'} E \otimes K_{w(s) \cdot (0, \lambda)} F \otimes A(-|\lambda|)$$

where  $P(r, s)$  is the set of partitions satisfying

$$\lambda_s \geq r + s \quad \lambda_{s+1} \leq s$$

and  $w(s) \cdot (0, \lambda) = (\lambda_1 - r, \dots, \lambda_s - r, s^r, \lambda_{s+1}, \dots, \lambda_t)$ .

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(Each term in the above resolution is built up as the direct sum of tensor products of Weyl modules  $K_\lambda E$  defined for any weight  $\lambda$  and any vector space  $E$ . We remark at the outset that some authors use the notation  $\mathbb{S}_\lambda E$  to denote the Weyl module of  $E$  associated to the partition  $\lambda$ .)

In analogy with the usual determinantal varieties, we will henceforth call the orbit closures in the remaining families (2)-(4) orthogonal or symplectic analogues of determinantal varieties. Consider a vector space  $E$  of dimension  $e$  and  $F$  a orthogonal (resp. symplectic) vector space of dimension  $f$  equipped with a symmetric (resp. skew-symmetric) non-degenerate bilinear product  $\langle \cdot, \cdot \rangle$  for which  $F$  can have isotropic spaces of maximal dimension. The non-degenerate bilinear form  $\langle \cdot, \cdot \rangle$  defines an isomorphism  $i : F \cong F^*$  by  $v \mapsto \langle v, \cdot \rangle$ . When we consider a linear map  $\phi : E \rightarrow F$  we define  $\phi^* : F \rightarrow E^*$  as the composition  $\phi^* \circ i$ . With these notations, all orthogonal (resp. symplectic) analogues of determinantal varieties are of the form:

$$\bar{O}_{r_1, r_2} = \{ \phi \in \text{Hom}(E, F) : \text{rank } \phi \leq r_1 \text{ and } \text{rank}(\phi^* \phi) \leq r_2 \}$$

where  $r_1$  and  $r_2$  are integers satisfying the compatibility conditions

$$(2) \quad 0 \leq r_2 \leq r_1 \leq e \quad \text{and} \quad 2r_1 - r_2 \leq f.$$

In this paper, we find non-minimal free resolutions for the coordinate rings for  $\bar{O}_{r_1, r_2}$  for the families (2)-(4).

This paper is organized as follows. In section 2, we use results from Józefiak, Pragacz and Weyman [4], to produce a relative version of a resolution of a rank variety for symmetric (resp. skew-symmetric) over a Grassmannian. Applying the geometric technique (see [12]) to this relative resolution, we obtain a double complex and in our main theorem 2.2 prove that the iterated mapping cone construction on this double complex is in fact a resolution for  $k[\bar{O}_{r_1, r_2}]$  as a quotient ring of  $k[\text{Hom}(E, F)]$ .

In section 3 we calculate the length of the resolution obtained in section 2, which allows us to define a necessary condition for when an orbit closure is Cohen-Macaulay. In Theorem 3.1, we prove that all orbit closures  $\bar{O}_{r_1, r_2}$  are Cohen-Macaulay except in the narrow case when  $F$  is orthogonal,  $f = 2r_1 - r_2$  and  $r_2 \neq 0$ . For this last class of orbit closures, we cannot conclude at present which are and which are not Cohen-Macaulay but we provide an example to show that not all of them are.

Because of the nature of the resolutions obtained in section 2, if  $\bar{O}_{r_1, r_2}$  is Cohen-Macaulay, it is an easy task to calculate the type of  $k[\bar{O}_{r_1, r_2}]$  as a  $k[\text{Hom}(E, F)]$ -module. Therefore, in section 4 we provide explicit formulas for the type and determine necessary and sufficient conditions for an orbit closure  $\bar{O}_{r_1, r_2}$  to be Gorenstein.

As it turns out, we do not need to investigate the behavior of orbits in families (2)-(4) separately. All the calculations can be done uniformly if we use the following numerical convention:

$$\varepsilon = \begin{cases} 1 & \text{if } F \text{ is an orthogonal space} \\ -1 & \text{if } F \text{ is a symplectic space} \end{cases}$$

## 2. RESOLUTION OF THE COORDINATE RINGS OF ORBIT CLOSURES

We assume the reader is familiar with the notion of a partition (see [11] for a reference) but since many of the objects we use are parametrized by partitions, we establish our notations before presenting the construction of our resolution. A *partition*  $\lambda$  is a finite sequence of positive integers  $(\lambda_1, \lambda_2, \dots, \lambda_t)$ . We call the *depth*  $l(\lambda)$  for the length  $t$  of the sequence and the *content*  $|\lambda|$  is the sum  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_t$ . The *conjugate*  $\lambda'$  is the partition defined by

$$\lambda'_j = \#\{i : \lambda_i \geq j\}.$$

We call the *rank* of a partition the length of the diagonal (when viewing a partition as stacked rows of boxes), i.e. the quantity

$$\#\{i : \lambda_i \geq i \text{ and } \lambda'_i \geq i\}.$$

The *Frobenius notation* describes a partition using two strictly decreasing sequences of positive integers with length the rank  $r$  of  $\lambda$  by writing

$$\lambda = (a_1, \dots, a_r | b_1, \dots, b_r)$$

with

$$\begin{cases} a_i = \lambda_i - i + 1 & \text{for } 1 \leq i \leq r \\ b_i = \lambda'_i - i + 1 & \text{for } 1 \leq i \leq r. \end{cases}$$

Again viewing the partition as stacked rows of boxes, the integers  $a_i$  (resp.  $b_i$ ) correspond to how many boxes are at or directly to the left (resp. below) the  $i$ 'th box on the diagonal.

As stated in the introduction, our idea for the construction of resolutions of  $k[\bar{O}_{r_1, r_2}]$  is to begin with a relative version of the resolution for the coordinate rings of the varieties  $\bar{O}_{e, r_2}$  over the Grassmannian  $\mathbb{G}_{e-r_1}(E)$ . Letting  $\mathcal{Q}$  be the tautological quotient bundle over  $\mathbb{G}_{e-r_1}(E)$ , we will create a resolution of  $\text{Sym}(\mathcal{Q} \otimes F)$ -modules of a particular ideal sheaf  $\mathcal{J}$  that corresponds to certain symmetric or antisymmetric maps of rank  $r_2$  in  $\mathcal{H}om(\mathcal{Q}, \mathcal{Q})$ . Applying the global section functor  $\Gamma(\mathbb{G}_{e-r_1}(E), -)$  to our resolution for  $\mathcal{J}$  leads to a complex of  $\text{Sym}(E \otimes F)$ -modules from which we produce a resolution for  $k[\bar{O}_{r_1, r_2}]$  as a quotient ring of  $k[\text{Hom}(E, F)]$ .

To carry out our approach, we need to possess resolutions for the coordinate rings of symmetric or antisymmetric matrices with specified rank. Józefiak, Pragacz and Weyman thoroughly studied such resolutions in [4] and we borrow their results. For completeness, provide a condensed description of the terms in the resolution, using a different though equivalent formulation that will mesh more easily with the results of this paper.

For the variety of antisymmetric matrices of a specified rank, we remark first that the rank must be even. The authors in [4] set  $G = k^n$ ,  $X = \text{Alt}_n(k)$  the affine space of antisymmetric matrices, and  $Y_{2p}$  the variety of all matrices of rank at most  $2p$ . The  $i$ 'th component of a minimal free resolution of  $k[Y_{2p}]$  over  $A = k[X]$  is equal to

$$(3) \quad \bigoplus_{d_s(\lambda)=i} K_\lambda G \otimes A(-|\lambda|)$$

where writing the partition  $\lambda = (a_1, \dots, a_t | b_1, \dots, b_t)$  in Frobenius notation,  $\lambda$  satisfies  $b_i = a_i + 2p + 1$  and  $d_s(\lambda) = \sum_{i=1}^t a_i$ .

For symmetric matrices, in [4] the authors set  $G = k^n$ ,  $X = \text{Sym}_n(k)$  the affine space of symmetric matrices, and  $Y_r$  is the variety of all matrices of rank at most  $r$ . They construct a minimal free resolution of  $k[Y_r]$  over  $A = k[X]$  whose  $i$ 'th component is equal to

$$(4) \quad \bigoplus_{d_o(\lambda)=i} K_\lambda G \otimes A(-|\lambda|)$$

where writing the partition  $\lambda = (a_1, \dots, a_t | b_1, \dots, b_t)$  in Frobenius notation,  $\lambda$  satisfies  $b_i = a_i + r - 1$ ,  $t$  is even and  $d_o(\lambda) = \sum_{i=1}^t (a_i - 1)$ .

We are now in a position to present our resolution of  $k[\bar{O}_{r_1, r_2}]$  for any  $\bar{O}_{r_1, r_2}$ , an orthogonal or symplectic analogue of a determinantal ideal. However, we first remind the reader of a proposition in homological algebra that plays a central role in our construction.

**Proposition 2.1.** *Let  $R$  be a ring. Consider a finite complex  $C_\bullet$  of  $R$ -modules*

$$0 \longrightarrow C_s \xrightarrow{\partial_s} C_{s-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

Suppose also that for each  $0 \leq i \leq s$  we have finite projective resolutions  $P_\bullet^{(i)} \rightarrow C_i \rightarrow 0$  augmented so that  $P_0^{(i)} = C_i$ . Then there exists a finite projective resolution

$$L_\bullet \rightarrow \text{Coker}(\partial_1) \rightarrow 0 \quad \text{such that} \quad L_i = \bigoplus_{j+k=i} P_k^{(j)}.$$

*Proof.* This proposition merely describes a standard iterated mapping cone construction so we leave it to the reader as an exercise in homological algebra. (See [2] for a reference.)  $\square$

With the iterated mapping cone construction at our disposal, we now present our construction of a resolution of  $k[\bar{O}_{r_1, r_2}]$  as a module over  $k[\text{Hom}(E, F)]$ . Let  $\mathbb{G}_{e-r_1}(E)$  be the Grassmannian of  $(e-r_1)$ -dimensional subspaces in  $E$ . We have a sequence of tautological bundles over  $\mathbb{G}_{e-r_1}(E)$ :

$$0 \rightarrow \mathcal{R} \rightarrow \mathbb{G}_{e-r_1}(E) \times E \rightarrow \mathcal{Q} \rightarrow 0$$

Over  $\mathbb{G}_{e-r_1}(E)$  define the sheaf of algebras  $\mathcal{A} = \text{Sym}(\mathcal{Q} \otimes F)$ . Let  $\mathcal{J}$  be the sheaf of ideals that corresponds to injective morphisms  $\phi: \mathcal{Q} \rightarrow F$  such that  $\text{rank } \phi^* \phi \leq r_2$  and consider a minimal free resolution of  $\mathcal{A}/\mathcal{J}$ :

$$(5) \quad 0 \rightarrow \mathcal{F}_m \rightarrow \dots \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{A} \rightarrow 0$$

Let  $\Gamma$  be the functor of sections over  $\mathbb{G}_{e-r_1}(E)$ . Calling  $\mathcal{F}_0 = \mathcal{A}$  for consistency, we obtain the complex:

$$0 \rightarrow \Gamma(\mathcal{F}_m) \rightarrow \dots \rightarrow \Gamma(\mathcal{F}_1) \rightarrow \Gamma(\mathcal{F}_0) \rightarrow 0$$

From formulas (3) and (4) we can calculate the terms  $\mathcal{F}_m$  as follows. Let  $P(a, l)$  be the set of partitions  $\lambda$  contained in the  $a \times (a-l)$ -rectangle such that if

$$\lambda = (a_1, \dots, a_r | b_1, \dots, b_r)$$

in Frobenius notation, then  $b_i = a_i + l$  for all  $1 \leq i \leq r$ . Let  $P(a, l)_{\text{even}}$  be the partitions of  $P(a, l)$  with even rank  $r$ . Define also the functions

$$d_s(\lambda) = \sum_{i=1}^r a_i \quad \text{and} \quad d_o(\lambda) = \sum_{i=1}^r (a_i - 1)$$

and the sheaves

$$\mathcal{M}_\lambda = K_\lambda \mathcal{Q} \otimes \mathcal{A}(-|\lambda|) \quad \text{with global sections} \quad M_\lambda = \Gamma(\mathcal{M}_\lambda).$$

Then the relative version of (3) and (4) over the Grassmannian gives

$$\mathcal{F}_j = \begin{cases} \bigoplus_{\lambda: d_o(\lambda)=j} \mathcal{M}_\lambda & \text{if } F \text{ is orthogonal} \\ \bigoplus_{\lambda: d_s(\lambda)=j} \mathcal{M}_\lambda & \text{if } F \text{ is symplectic} \end{cases}$$

and

$$\Gamma(\mathcal{F}_j) = \begin{cases} \bigoplus_{\lambda: d_o(\lambda)=j} M_\lambda & \text{if } F \text{ is orthogonal} \\ \bigoplus_{\lambda: d_s(\lambda)=j} M_\lambda & \text{if } F \text{ is symplectic} \end{cases}$$

where the summations are taken over  $\lambda \in P(r_1, r_2 - \varepsilon)_{\text{even}}$  if  $F$  is orthogonal and over  $\lambda \in P(r_1, r_2 - \varepsilon)$  if  $F$  is symplectic. Furthermore, we know from [4] that the length of the resolution (5) is

$$m = \frac{(r_1 - r_2)(r_1 - r_2 + \varepsilon)}{2}.$$

We now need to find resolutions for the  $A$ -modules  $\Gamma(\mathcal{F}_j)$ . We use what is generically called the geometric technique and refer the reader to chapter 5 of [12] for a thorough presentation of the technique. Using the geometric technique with the bundle  $\xi = \mathcal{R} \otimes F$ , we will prove below in

Lemma 2.3 that the complex  $F_\bullet(\mathcal{M}_\lambda)$  is an augmented resolution of  $M_\lambda$  as an  $A$ -module, where  $A = k[\text{Hom}(E, F)]$ . Consequently, we obtain the diagram of  $A$ -modules

$$(6) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Gamma(\mathcal{F}_m) & \longrightarrow & \cdots & \longrightarrow & \Gamma(\mathcal{F}_1) & \longrightarrow & \Gamma(\mathcal{F}_0) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ & & \bigoplus_m F_\bullet(\mathcal{M}_\lambda) & & \bigoplus_1 F_\bullet(\mathcal{M}_\lambda) & & \bigoplus_0 F_\bullet(\mathcal{M}_\lambda) & & & & \end{array}$$

where the row across the top is a complex, the columns are augmented resolutions and  $\bigoplus_j F_\bullet(\mathcal{M}_\lambda)$  indicates that the direct sum of complexes is taken over all  $\lambda$  in  $P(r_1, r_2 - \varepsilon)_{\text{even}}$  (resp.  $P(r_1, r_2 - \varepsilon)$ ) such that  $d_o(\lambda) = j$  (resp.  $d_s(\lambda) = j$ ). We can therefore use Proposition 2.1.

For each  $\lambda \in P(r_1, r_2 - \varepsilon)$ , we need to study the complex  $F_\bullet(\mathcal{M}_\lambda)$  with  $\xi = \mathcal{R} \otimes F$ . We are interested in the length of the complex  $F_\bullet(\mathcal{M}_\lambda)$  and whether any terms appear in negative degrees.

By Theorem 5.1.2 in [12], if we set  $A = k[\text{Hom}(E, F)] \cong \text{Sym}(E \otimes F)$ , then

$$(7) \quad F_i(\mathcal{M}_\lambda) = \bigoplus_{j \geq 0} H^j \left( \mathbb{G}_{e-r_1}(E), \bigwedge^{i+j} (\mathcal{R} \otimes F) \otimes K_\lambda \mathcal{Q} \right) \otimes_k A(-i-j)$$

$$(8) \quad = \bigoplus_{j \geq 0} \bigoplus_{\mu: |\mu|=i+j} K_\mu F \otimes H^j(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{\lambda, \mu}) \otimes_k A(-i-j)$$

where  $\mathcal{V}_{\lambda, \mu}$  is the bundle  $\mathcal{V}_{\lambda, \mu} = K_\lambda \mathcal{Q} \otimes K_\mu \mathcal{R}$  over the Grassmannian  $\mathbb{G}_{e-r_1}(E)$ . Consequently, in order to study the terms of the complex  $F_\bullet(\mathcal{M}_\lambda)$ , we must calculate the cohomology groups

$$H^j(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{\lambda, \mu})$$

where  $\lambda \in P(r_1, r_2 - \varepsilon)$  if  $F$  is symplectic,  $\lambda \in P(r_1, r_2 - \varepsilon)_{\text{even}}$  if  $F$  is orthogonal, and  $\mu$  is in the rectangle of dimensions  $(e - r_1) \times f$ .

With  $\lambda$  fixed, by Bott's Theorem (see Theorem 4.1.8 [12] for a reference), two cases may occur:

a) if there exists  $\sigma \in \mathcal{W}_E = S_e$  not the identity such that  $\sigma^\bullet((\lambda, \mu)) = (\lambda, \mu)$ , then

$$H^j(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{\lambda, \mu}) = 0 \quad \text{for all } \mu$$

b) if there exists  $\sigma \in S_e$  such that  $\sigma^\bullet((\lambda, \mu))$  is a dominant weight, then

$$H^j(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{\lambda, \mu}) = \begin{cases} 0 & \text{if } j \neq l(\sigma) \\ K_{\sigma^\bullet((\lambda, \mu))} E & \text{if } j = l(\sigma) \end{cases}$$

Consequently, with a fixed partition  $\lambda$ , we must determine the following set

$$(9) \quad D_\lambda = \{ \mu \in (e - r_1) \times f\text{-rectangle} \mid \exists \sigma \in S_e - \{id\} \text{ such that } \sigma^\bullet((\lambda, \mu)) \text{ is dominant} \}.$$

By Bott's Theorem, each  $\mu \in D_\lambda$  contributes one term to the resolution  $F_\bullet(\mathcal{M}_\lambda)$  which appears in degree

$$(10) \quad \begin{aligned} d_\lambda(\mu) &= |\sigma^\bullet((\lambda, \mu))| - |\lambda| - l(\sigma) \\ &= |\mu| - l(\sigma) \end{aligned}$$

where  $\sigma^\bullet((\lambda, \mu))$  is a dominant weight. (From this last expression, it might appear as though the degree does not depend on  $\lambda$  but that would be incorrect since  $\sigma$  depends on both  $\mu$  and  $\lambda$ .) Even if we do not determine the set of partitions  $D_\lambda$  completely, we will still profit from knowing extremal values of the function  $d_\lambda : D_\lambda \rightarrow \mathbb{Z}$ .

The above construction leads to this paper's main theorem.

**Theorem 2.2.** *With notations as above, the  $m$ 'th iterated mapping cone construction on the complex*

$$0 \longrightarrow \Gamma(\mathcal{F}_m) \longrightarrow \cdots \longrightarrow \Gamma(\mathcal{F}_1) \longrightarrow \Gamma(\mathcal{F}_0) \longrightarrow 0$$

*is a resolution of  $k[\bar{O}_{r_1, r_2}]$  as a  $k[\mathrm{Hom}(E, F)]$ -module. More explicitly, there is a free resolution*

$$L_\bullet \longrightarrow k[\mathrm{Hom}(E, F)] \longrightarrow 0$$

*of  $k[\bar{O}_{r_1, r_2}]$  as a  $k[\mathrm{Hom}(E, F)]$ -module with*

$$L_i = \bigoplus_{l+k=i} \bigoplus_{d(\lambda)=l} \bigoplus_{j \geq 0} \bigoplus_{\mu: |\mu|=k+j} K_{\mu'} F \otimes H^j(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{\lambda, \mu}) \otimes A(-j-k)$$

*where in the second summation  $d(\lambda)$  refers to  $d_o(\lambda)$  and  $\lambda \in P(r_1, r_2 - \varepsilon)_{\text{even}}$  if  $F$  is orthogonal and  $d(\lambda) = d_s(\lambda)$  and  $\lambda \in P(r_1, r_2 - \varepsilon)$  if  $F$  is symplectic. Furthermore, one calculates the terms  $H^j(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{\lambda, \mu})$  using Bott's Theorem as described above.*

*Proof.* The proof of this theorem follows from the next two lemmas.  $\square$

**Lemma 2.3.** *The complex  $F_\bullet(\mathcal{M}_\lambda)$  is a resolution of  $M_\lambda = \Gamma(K_\lambda \mathcal{Q} \otimes \mathcal{A}(-|\lambda|))$  as a  $k[\mathrm{Hom}(E, F)]$  module. In particular, if we let  $d_\lambda : D_\lambda \longrightarrow \mathbb{Z}$  be the degree function as above, then  $d_\lambda(\mu) \geq 0$  for all  $\mu \in D_\lambda$  and if  $d_\lambda(\mu) = 0$  then  $\mu = (0, 0, \dots, 0)$ .*

*Proof.* Consider the determinantal variety  $Y = \{\phi \in \mathrm{Hom}(E, F) : \mathrm{rank} \phi \leq r_1\}$  along with one of its standard desingularizations

$$Z = \{(\phi, R) \in \mathrm{Hom}(E, F) \times \mathbb{G}_{e-r_1}(E) : \phi|_R = 0\}.$$

Over the Grassmannian  $V = \mathbb{G}_{e-r_1}(E)$  one has the tautological sequence of vector bundles

$$0 \longrightarrow \mathcal{R} \longrightarrow E \longrightarrow \mathcal{Q} \longrightarrow 0$$

We then see that  $Z$  is the total space of the vector bundle  $\mathcal{Q}^* \otimes F$  and hence in the setup of the geometric technique we utilize the bundles

$$\begin{aligned} \eta &= \mathcal{Q} \otimes F^* \\ \xi &= \mathcal{R} \otimes F^* . \end{aligned}$$

We employ the set up of Theorem 5.1.2 in [12] with the diagram

$$\begin{array}{ccc} Z & \hookrightarrow & \mathbb{G}_{e-r_1}(E) \times \mathrm{Hom}(E, F) \\ q' \downarrow & & \downarrow q \\ Y & \hookrightarrow & \mathrm{Hom}(E, F) \end{array}$$

We deduce that

$$H_{-i}(F_\bullet(K_\lambda \mathcal{Q})) = H^i(\mathbb{G}_{e-r_1}(E), K_\lambda \mathcal{Q} \otimes \mathrm{Sym}(\mathcal{Q} \otimes F)) .$$

However, by the straightening rule

$$\mathrm{Sym}^m(\mathcal{Q} \otimes F) = \bigoplus_{\mu: |\mu|=m} K_\mu \mathcal{Q} \otimes K_\mu F$$

and Littlewood-Richardson rule, it is not hard to show that

$$H^i(\mathbb{G}_{e-r_1}(E), K_\lambda \mathcal{Q} \otimes \mathrm{Sym}(\mathcal{Q} \otimes F)) = \begin{cases} K_\lambda E & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Therefore, the complex  $F_\bullet(\mathcal{M}_\lambda)$  is in fact a resolution of  $M_\lambda$  as a  $k[\mathrm{Hom}(E, F)]$  module.

The rest of the lemma follows immediately.  $\square$

**Lemma 2.4.** *Consider the diagram in equation 6. Set  $A = k[\text{Hom}(E, F)]$ . We have the following equality of  $A$ -modules:*

$$\text{Coker}(\Gamma(\mathcal{F}_1) \rightarrow \Gamma(\mathcal{F}_0)) = k[\bar{O}_{r_1, r_2}].$$

*Proof.* Let us begin by determining the first few terms that resolve  $M = \text{Coker}(\Gamma(\mathcal{F}_1) \rightarrow \Gamma(\mathcal{F}_0))$ . From the iterated mapping cone construction we form a resolution of  $M$  which we call  $L_\bullet$ . We can easily obtain the terms  $L_0$  and  $L_1$  from our construction and Proposition 2.1.

$L_0 = \Gamma(\mathcal{F}_0)$  where  $\mathcal{F}_0 = \mathcal{A} = \text{Sym}(\mathcal{Q} \otimes F)$ . Thus by Bott's Theorem over the Grassmanian, whether  $F$  is orthogonal or symplectic, we obtain  $L_0 = A = k[\text{Hom}(E, F)]$ .

For  $L_1$ , we recall first that by Proposition 2.1 that  $L_1 = \Gamma(\mathcal{F}_1) \oplus P_1^{(0)}$  where  $P_\bullet^{(0)} \rightarrow \Gamma(\mathcal{F}_0)$  is an augmented resolution of  $\Gamma(\mathcal{F}_0)$  as an  $A$  module. We notice first that

$$\mathcal{F}_1 = \begin{cases} K_{(1^{r_2+2})} \mathcal{Q} \otimes \mathcal{A}(-r_2 - 2) & \text{if } F \text{ is symplectic} \\ K_{(2^{r_2+1})} \mathcal{Q} \otimes \mathcal{A}(-2r_2 - 2) & \text{if } F \text{ is orthogonal} \end{cases}$$

and consequently that

$$\Gamma(\mathcal{F}_1) = \begin{cases} \bigwedge^{r_2+2} E \otimes \mathcal{A}(-r_2 - 2) & \text{if } F \text{ is symplectic} \\ K_{(2^{r_2+1})} E \otimes \mathcal{A}(-2r_2 - 2) & \text{if } F \text{ is orthogonal} . \end{cases}$$

As for calculating  $P_1^{(0)}$ , we use the formula 8 with  $\lambda = 0$ . We can rewrite (8) in this case as follows:

$$P_1^{(0)} = \bigoplus_{\mu: |\mu| \geq 1} K_\mu F \otimes H^{|\mu|-1}(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{(0^{r_1}, \mu)}) \otimes A(|\mu|)$$

If there exists a permutation in the Weyl group of  $\mathcal{W} = S_e$  that sends  $(0^{r_1}, \mu)$  to a dominant weight via the dotted action, we call this permutation  $\sigma_\mu$ . By Bott's Theorem over the usual Grassmannian, if  $\sigma_\mu$  exists for some  $\mu$  then  $H^{|\mu|-1}(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{(0^{r_1}, \mu)}) \neq 0$  only if  $l(\sigma_\mu) = |\mu| - 1$ . It's a simple exercise to see that only the partition  $\mu = (r_1 + 1, 0, \dots, 0)$  satisfies this requirement and that

$$H^{|\mu|-1}(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{(0^{r_1}, \mu)}) = \bigwedge^{r_1+1} E .$$

Consequently, we deduce that

$$L_1 = \begin{cases} \bigwedge^{r_2+2} E \otimes \mathcal{A}(-r_2 - 2) \oplus \bigwedge^{r_1+1} E \otimes \bigwedge^{r_1+1} F \otimes \mathcal{A}(-r_1 - 1) & \text{if } F \text{ is symplectic} \\ K_{(2^{r_2+1})} E \otimes \mathcal{A}(-2r_2 - 2) \oplus \bigwedge^{r_1+1} E \otimes \bigwedge^{r_1+1} F \otimes \mathcal{A}(-r_1 - 1) & \text{if } F \text{ is orthogonal} . \end{cases}$$

The results from [4] and known results from resolutions of determinantal varieties, we identify  $\text{Im}(L_1 \rightarrow L_0)$  with an ideal  $I \subset A$  generated by  $(r_2 + 1)$ -minors of symmetric matrices  $E \rightarrow E^*$  if  $F$  is orthogonal (resp.  $(r_2 + 2)$ -pfaffians on antisymmetric matrices  $E \rightarrow E^*$  if  $F$  is symplectic) and  $(r_1 + 1)$ -minors of maps in  $\text{Hom}(E, F)$ .

In order to finish proving the lemma, we must show that the ideal  $I$  is radical, where  $\text{Coker}(L_1 \rightarrow L_0) = A/I$ . Let us suppose that  $\sqrt{I} \neq I$ . Notice that  $I$  is equivariant with respect to the group  $GL(E) \times O(F)$  (resp.  $GL(E) \times Sp(F)$ ) and hence so is  $\sqrt{I}$ . Let  $V$  be an irreducible representation of  $GL(E) \times O(F)$  (resp.  $GL(E) \times Sp(F)$ ) in  $\sqrt{I}$  but not in  $I$ . Then  $V$  is contained in

$$\bigoplus_{\lambda: |\lambda| \leq r_1} K_\lambda E \otimes K_\lambda F$$

because otherwise  $V$  would be contained in the ideal generated by  $(r_1 + 1)$ -minors of maps in  $\text{Hom}(E, F)$ . But this means that the highest weight vector of  $V$  already occurs in  $\text{Sym}(E' \otimes F)$  where  $\dim E' = r_1$ . In otherwords, it is enough to show that the ideal  $I$  is reduced in the case where  $\dim E = r_1$ . However, this follows as a consequence of the main results in [4] where the

authors calculate resolutions of the coordinate rings of the space of symmetric (resp. anti-symmetric) matrices of rank  $r_2$ . The authors' resolutions are precisely

$$0 \longrightarrow \Gamma(\mathcal{F}_m) \longrightarrow \dots \longrightarrow \Gamma(\mathcal{F}_1) \longrightarrow \Gamma(\mathcal{F}_0) \longrightarrow 0$$

when we assume that  $\dim E = r_1$ . Therefore, the ideal  $I$  is radical and the lemma follows.  $\square$

### 3. WHICH ORBIT CLOSURES ARE COHEN-MACAULAY?

In Corollary 2.4 of [10], the author proved the following codimension formula for the orbit closures  $\bar{O}_{r_1, r_2}$  in  $\text{Hom}(E, F)$ ,

$$(11) \quad \text{codim } \bar{O}_{r_1, r_2} = (e - r_1)(f - r_1) + \frac{(r_1 - r_2)(r_1 - r_2 + \varepsilon)}{2}.$$

Comparing this formula to the lengths of the resolutions obtained in Theorem 2.2 and in most cases determine whether or not  $\bar{O}_{r_1, r_2}$  is Cohen Macaulay.

**Theorem 3.1.** *Let  $E$  be a vector space of dimension  $e$  and  $F$  a symplectic (resp. orthogonal) space of dimension  $f$ . Let  $r_1$  and  $r_2$  be any ranks that satisfy the compatibility conditions (2). If it is not the case that  $F$  is orthogonal,  $f = 2r_1 - r_2$  and  $r_2 \neq 0$ , then the orbit closure  $\bar{O}_{r_1, r_2}$  is Cohen-Macaulay.*

Before proving this theorem, we need to establish two lemmas about the dotted Weyl group action in the situation of this theorem.

**Lemma 3.2.** *Let  $d_\lambda : D_\lambda \rightarrow \mathbb{Z}$  be the degree function as in (10). Then*

$$\max_{\mu \in D_\lambda} \{d_\lambda(\mu)\} = d_\lambda(\mu_{max})$$

where  $\mu_{max}$  is the partition  $\mu \in D_\lambda$  such that  $|\mu|$  is largest.

*Proof.* For any  $\sigma \in S_e$ , define the set  $U_\sigma = \{\mu \in D_\lambda : \sigma^\bullet((\lambda, \mu)) \text{ is dominant}\}$ . Clearly

$$\max_{\mu \in U_\sigma} \{d_\lambda(\mu)\} = d_\lambda(\mu')$$

where  $\mu'$  is such that  $|\mu'| = \max_{\mu \in U_\sigma} \{|\mu|\}$ .

Now let  $\sigma' = \tau\sigma$  where  $\tau$  is a transposition such that  $l(\sigma') = l(\sigma) + 1$ . If  $U_{\sigma'} \neq \emptyset$ , then

$$\max_{\mu \in U_\sigma} \{|\mu|\} < \max_{\mu \in U_{\sigma'}} \{|\mu|\}.$$

However, there exists a unique shuffle  $\sigma_{max} \in S_e$  with  $l(\sigma_{max})$  largest possible such that  $U_{\sigma_{max}} \neq \emptyset$ . Therefore

$$\begin{aligned} \max_{\mu \in D_\lambda} \{d_\lambda(\mu)\} &= d_\lambda(\mu') && \text{where } \mu' \text{ is such that } |\mu'| = \max_{\mu \in U_{\sigma_{max}}} \{|\mu|\} \\ &= d_\lambda(\mu_{max}) && \text{where } \mu_{max} \text{ is such that } |\mu_{max}| = \max_{\mu \in D_\lambda} \{|\mu|\}. \end{aligned}$$

$\square$

**Lemma 3.3.** *Set  $A = k[\text{Hom}(E, F)]$ . Let  $\lambda$  be a partition in the rectangle  $((r_1 - r_2 + \varepsilon)^{r_1})$ , and let us consider  $M_\lambda = \Gamma(K_\lambda \mathcal{Q} \otimes \mathcal{A}(-|\lambda|))$  as an  $A$ -module. One of the following two cases occurs:*

a) *If  $f - \lambda_1 - r_1 \geq 0$ , then the projective dimension of  $M_\lambda$  is*

$$\text{pd}_A(M_\lambda) = (e - r_1)(f - r_1).$$

b) *If  $f - \lambda_1 - r_1 = -1$ , then the projective dimension of  $M_\lambda$  is*

$$\text{pd}_A(M_\lambda) = (e - r_1)(f - r_1 + 1).$$

Furthermore, the second case only occurs when  $\varepsilon = 1$  (i.e. in the orthogonal case),  $f = 2r_1 - r_2$  and  $\lambda_1 = r_1 - r_2 + \varepsilon$ .

*Proof.* The technique of the proof is to refer to Lemma 3.2 and find  $\mu_{\max}$ , the largest  $\mu \in D_\lambda$  such that there exists a  $\sigma \in S_e$  such that  $\sigma^\bullet((\lambda, \mu))$  is a dominant weight. In particular,

$$\text{pd}_A(M_\lambda) = d_\lambda(\mu_{\max}) = |\mu_{\max}| - l(\sigma).$$

Let  $1 \leq i_1 < i_2 < \cdots < i_k = r_1$  be the indices where  $\lambda$  has jumps. More precisely, we define the increasing sequence  $1 \leq i_1 < i_2 < \cdots < i_k = r_1$  by the conditions that for all  $1 \leq l \leq k$  and all  $i_{l-1} < i'_l \leq i_l$ ,

$$\lambda_{i'_l} = \lambda_{i_l} \quad \text{and} \quad \lambda_{i_l} > \lambda_{i_{l+1}}.$$

Set  $n = \max\{0, f - \lambda_1 - r_1\}$ . We remark that since  $\lambda$  is in the rectangle  $((r_1 - r_2 + \varepsilon)^{r_1})$  the compatibility conditions (2) indicate that  $n \neq f - \lambda_1 - r_1$  only when  $f - \lambda_1 - r_1 = -1$  which only occurs when  $f = 2r_1 - r_2$ ,  $\varepsilon = 1$  and  $\lambda_1 = r_1 - r_2 + \varepsilon$ .

We construct the partition  $\mu = \mu_{\max}$  by looking at the jumps in  $\lambda$ . The dotted action of a Weyl element will allow up to  $n$  entries of  $\mu$  to get shifted to the left of all entries of  $\lambda$ . At each jump, one can insert at most  $\lambda_{i_l} - \lambda_{i_{l+1}}$  entries of  $\mu$ . Consequently, we obtain  $\mu$  by

$$\begin{array}{ll} 1 \leq i \leq n & \mu_i = f \\ n < i \leq n + \lambda_{i_1} - \lambda_{i_2} & \lambda_{i_1} + n = \mu_i - (r_1 - i_1) \\ n + \lambda_{i_1} - \lambda_{i_2} < i \leq n + \lambda_{i_1} - \lambda_{i_3} & \lambda_{i_2} + n + \lambda_{i_1} - \lambda_{i_2} = \mu_i - (r_1 - i_2) \\ n + \lambda_{i_1} - \lambda_{i_3} < i \leq n + \lambda_{i_1} - \lambda_{i_4} & \lambda_{i_3} + n + \lambda_{i_1} - \lambda_{i_3} = \mu_i - (r_1 - i_3) \\ \vdots & \vdots \end{array}$$

Remarking that  $\lambda_{i_1} = \lambda_1$  we rewrite this as

$$\begin{array}{ll} 1 \leq i \leq n & \mu_i = f \\ n < i \leq n + \lambda_{i_1} - \lambda_{i_2} & \mu_i = \lambda_1 + n + r_1 - i_1 \\ n + \lambda_{i_1} - \lambda_{i_2} < i \leq n + \lambda_{i_1} - \lambda_{i_3} & \mu_i = \lambda_1 + n + r_1 - i_2 \\ n + \lambda_{i_1} - \lambda_{i_3} < i \leq n + \lambda_{i_1} - \lambda_{i_4} & \mu_i = \lambda_1 + n + r_1 - i_3 \\ \vdots & \vdots \end{array}$$

But we must have  $i \leq |\mu| = e - r_1$  so these intervals over which we have defined  $\mu$  stop at the index  $j$  where

$$(12) \quad n + \lambda_1 - \lambda_{i_j} < e - r_1 \leq n + \lambda_1 - \lambda_{i_{j+1}}.$$

Thus for the indices  $n + \lambda_1 - \lambda_{i_j} < i \leq e - r_1$  we set  $\mu_i = \lambda_1 + n + r_1 - i_j$ .

Therefore we obtain

$$\begin{aligned} |\mu_{\max}| &= fn + (\lambda_{i_1} - \lambda_{i_2})(\lambda_1 + n + r_1 - i_1) + (\lambda_{i_2} - \lambda_{i_3})(\lambda_1 + n + r_1 - i_2) + \cdots \\ &\quad + (\lambda_{i_{j-1}} - \lambda_{i_j})(\lambda_1 + n + r_1 - i_{j-1}) + (e - r_1 - n - \lambda_1 + \lambda_{i_j})(\lambda_1 + n + r_1 - i_j). \end{aligned}$$

Furthermore, it is not too hard to see that

$$l(\sigma) = \sum_{i,j} \#\{(i,j) : \lambda_i + e - i < \mu_j + e - r_1 - j\}$$

and by our construction of  $\mu$  this is equal to

$$\begin{aligned} l(\sigma) &= nr_1 + (\lambda_{i_1} - \lambda_{i_2})(r_1 - i_1) + (\lambda_{i_2} - \lambda_{i_3})(r_1 - i_2) + \cdots \\ &\quad + (\lambda_{i_{j-1}} - \lambda_{i_j})(r_1 - i_{j-1}) + (e - r_1 - n - \lambda_1 + \lambda_{i_j})(r_1 - i_j). \end{aligned}$$

Consequently, we obtain

$$(13) \quad d_\lambda(\mu_{\max}) = |\mu_{\max}| - l(\sigma) \\ = n(f - r_1) + (\lambda_1 - \lambda_{i_j})(\lambda_1 + n) + (e - r_1 - n - \lambda_1 + \lambda_{i_j})(\lambda_1 + n)$$

$$(14) \quad = n(f - r_1) + (e - r_1 - n)(\lambda_1 + n).$$

A perhaps surprising remark to this equality is that  $d_\lambda(\mu_{\max})$  does not depend on the jumps in  $\lambda$  or the indices  $i_l$ .

At this point, we consider two cases depending on the value of  $n$ .

**Case 1:** Suppose that  $f - \lambda_1 - r_1 \geq 0$  which is false only when  $f = 2r_1 - r_2$ ,  $\varepsilon = 1$  and  $\lambda_1 = r_1 - r_2 + 1$ . In this case,  $n = f - \lambda_1 - r_1$ . When we replace  $n = f - \lambda_1 - r_1$  in equation (14), we obtain the following expected result after a few manipulations:

$$d_\lambda(\mu_{\max}) = (e - r_1)(f - r_1).$$

**Case 2:** Suppose that  $f - \lambda_1 - r_1 = -1$  which is true only when  $f = 2r_1 - r_2$ ,  $\varepsilon = 1$  and  $\lambda_1 = r_1 - r_2 + 1$ . In this case,  $n = 0 = f - \lambda_1 - r_1 + 1$ . When we replace  $n = f - \lambda_1 - r_1 + 1$  in equation (14), we obtain the following expression after a few manipulations:

$$d_\lambda(\mu_{\max}) = (f - \lambda_1 - r_1 + 1)(f - r_1) + (e - f + \lambda_1 - 1)(f - r_1 + 1) \\ = (e - r_1)(f - r_1 + 1).$$

□

*Proof. of Theorem 3.1*

Suppose first that  $F$  is orthogonal (i.e.  $\varepsilon = 1$ ) and  $f = 2r_1 - r_2$ . Theorems 4.9 and 4.10 in [10] establish that given this assumption  $\bar{O}_{r_1, r_2}$  is Cohen-Macaulayness when  $r_2 = 0$ . Notice that if we assume  $\dim F = 2r_1 - r_2$ ,  $r_2 = 0$  implies that  $F$  is even dimensional.

Suppose now that  $\varepsilon \neq 1$  or  $f > 2r_1 - r_2$ . By Proposition 3.3, the resolution  $L_\bullet$  created by the iterated mapping cone construction has length

$$(e - r_1)(f - r_1) + \frac{(r_1 - r_2)(r_1 - r_2 + \varepsilon)}{2}$$

which according to (11) is precisely the codimension of  $\bar{O}_{r_1, r_2}$ .

Even though  $L_\bullet$  is not a minimal resolution, a minimal resolution of  $k[\bar{O}_{r_1, r_2}]$  must have length less than or equal to  $(e - r_1)(f - r_1) + (r_1 - r_2)(r_1 - r_2 + \varepsilon)/2$ . However, by standard methods of algebraic geometry, a minimal resolution cannot be shorter than  $\text{codim } \bar{O}_{r_1, r_2}$  and hence it must have this length. This proves that  $\bar{O}_{r_1, r_2}$  is Cohen-Macaulay. □

Theorem 3.1 singles out the rather narrow case of orbits  $\bar{O}_{r_1, r_2}$  that satisfy  $F$  orthogonal,  $f = 2r_1 - r_2$  and  $r_2 \neq 0$ . Currently, in this article, we are not able to establish which of these orbits are not Cohen-Macaulay. However, we cannot hope that all orbits are Cohen-Macaulay and that our methods just aren't subtle enough to prove them so because, as the following example shows that some orbits in this narrow case are not Cohen-Macaulay.

**Example 3.4.** Let us consider  $F$  orthogonal,  $\dim E = 4$ ,  $\dim F = 5$ ,  $r_1 = 3$  and  $r_2 = 1$ . The codimension of the orbit closure is given by

$$\text{codim } \bar{O}_{r_1, r_2} = (e - r_1)(f - r_1) + \frac{(r_1 - r_2)(r_1 - r_2 + 1)}{2} = 5$$

On the other hand, the iterated mapping cone construction leads to the following resolution of the coordinate ring  $k[\bar{O}_{r_1, r_2}]$  as an  $A$ -modules, where  $A = k[\text{Hom}(E, F)]$ .

$$\begin{array}{c}
 0 \\
 \downarrow \\
 K_{(3^4)}E \otimes \bigwedge^4 F \otimes A(-12) \\
 \downarrow \\
 K_{(3^2, 2^2)}E \otimes \bigwedge^2 F \otimes A(-10) \oplus K_{(3^3, 2)}E \otimes \bigwedge^5 F \otimes A(-11) \\
 \downarrow \\
 K_{(3^2, 2, 1)}E \otimes F \otimes A(-9) \oplus K_{(3, 2^3)}E \otimes \bigwedge^3 F \otimes A(-9) \\
 \downarrow \\
 K_{(3^2, 2)}E \otimes A(-8) \oplus K_{(3, 2, 1^2)}E \otimes F \otimes A(-7) \oplus K_{(2^3, 1)}E \otimes \bigwedge^3 F \otimes A(-7) \\
 \downarrow \\
 K_{(3, 2, 1)}E \otimes A(-6) \oplus K_{(2^2, 1^2)}E \otimes F \otimes A(-6) \oplus K_{(2, 1^3)}E \otimes \bigwedge^5 F \otimes A(-5) \\
 \downarrow \\
 K_{(2^2)}E \otimes A(-4) \oplus \bigwedge^4 E \otimes \bigwedge^4 F \otimes A(-4) \\
 \downarrow \\
 A \\
 \downarrow \\
 0
 \end{array}$$

It is clear that no terms in this resolution can cancel, even when we decompose  $\bigwedge^i F$  to indecomposable orthogonal representations, and hence this resolution is already minimal. However, since it has length 6 and in particular is strictly greater than  $\text{codim } \bar{O}_{r_1, r_2}$ , the orbit closure is not Cohen-Macaulay.

#### 4. CALCULATIONS OF TYPE

As an application of the techniques presented in the above subsection, we propose to calculate the type of the orbit closure  $\bar{O}_{r_1, r_2}$  which is an important algebraic invariant. We state the definition given in [1].

**Definition 4.1.** Let  $(R, \mathfrak{m}, k)$  be a local Noetherian ring and  $M$  a finite  $R$ -module of depth  $t$ . The number  $r(M) = \dim_k \text{Ext}_R^t(k, M)$  is called the *type* of  $M$ .

We work in the ring  $A = k[\text{Hom}(E, F)]$  and study the module  $M = k[\bar{O}_{r_1, r_2}]$ . For every maximal ideal  $\mathfrak{m} \in \text{Supp } M$ ,  $A_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} = k$  and  $\text{depth } M_{\mathfrak{m}}$  is unchanged as is  $\dim_k \text{Ext}_{A_{\mathfrak{m}}}^t(k, M_{\mathfrak{m}})$ . We also call this common number the type of  $M = k[\bar{O}_{r_1, r_2}]$  and write it  $r(\bar{O}_{r_1, r_2})$ . We can easily show that  $r(\bar{O}_{r_1, r_2})$  is the dimension of the top free module in a minimal resolution of  $k[\bar{O}_{r_1, r_2}]$  as an  $A$  module.

In the previous subsection where we constructed the free resolution  $L_{\bullet}$  of  $k[\bar{O}_{r_1, r_2}]$  as an  $A$ -module, we did not obtain a minimal resolution. However, in every case except when  $F$  is orthogonal with

$f = 2r_1 - r_2$ , we obtained a resolution of depth equal to  $\text{codim } \bar{O}_{r_1, r_2}$  with a single term of the form  $K_\lambda E \otimes K_\mu F \otimes A(-|\lambda|)$  in the top term. Consequently, in a minimal resolution of  $k[\bar{O}_{r_1, r_2}]$ , the term of maximal degree is equal to the term of maximal degree we obtained by our iterated mapping cone construction. Let us prosaically call this term  $\text{Top}(\bar{O}_{r_1, r_2})$ . Therefore we can calculate  $r(\bar{O}_{r_1, r_2}) = \dim \text{Top}(\bar{O}_{r_1, r_2})$  fairly easily.

**Proposition 4.2.** *Let  $E$  be a vector space of dimension  $e$  and  $F$  be a symplectic space of dimension  $f$ . Consider the orbit closure  $\bar{O}_{r_1, r_2}$ . One of the three following cases occurs:*

a) *If  $f - e - r_1 + r_2 + 1 < 0$  then*

$$r(\bar{O}_{r_1, r_2}) = \prod_{\substack{1 \leq i \leq f - r_1 \\ 1 \leq j \leq r_1}} \frac{e - r_2 + j - i - 1}{f - r_1 + j - i}.$$

b) *If  $f - e - r_1 + r_2 + 1 = 0$  then*

$$r(\bar{O}_{r_1, r_2}) = 1.$$

c) *If  $f - e - r_1 + r_2 + 1 > 0$  then*

$$r(\bar{O}_{r_1, r_2}) = \prod_{\substack{1 \leq i \leq e - r_1 \\ 1 \leq j \leq r_1}} \frac{f - 2r_1 + r_2 + j - i + 1}{e - r_1 + j - i}.$$

*Proof.* Note that since  $F$  is symplectic, we have  $\varepsilon = -1$ .

We determine  $\text{Top}(\bar{O}_{r_1, r_2})$  as  $F_{\max}(K_{\lambda_{\max}} \mathcal{Q})$  in the same way we determined projective dimension of  $F_\bullet(K_{\lambda_{\max}} \mathcal{Q})$  in Proposition 3.3. We know that when  $F$  is symplectic  $\lambda_{\max} = ((r_1 - r_2 - 1)_1^r)$ . We now need to calculate the partition  $\mu$  such that

$$F_{\max}(K_{\lambda_{\max}} \mathcal{Q}) = H^{l(\sigma)}(\mathbb{G}_{e-r_1}(E), \mathcal{V}_{\lambda, \mu}) \otimes K_{\mu'} F.$$

In order to do so, we must consider three separate cases.

**Case 1:**  $f - e - r_1 + r_2 + 1 < 0$ .

Following the calculations in Proposition 3.3, we obtain

$$\mu = (f^{(f-2r_1+r_2+1)}, (f-r_1)^{(e-f+r_1-r_2-1)}).$$

Furthermore, a simple check yields  $\sigma^\bullet((\lambda, \mu)) = ((f-r_1)^e)$  which leads to

$$r(\bar{O}_{r_1, r_2}) = \dim K_{(f-r_1)^e} E \otimes K_{\mu'} F = \dim K_{\mu'} F$$

where  $\mu' = ((e-r_1)^{(f-r_1)}, (f-2r_1+r_2+1)^{r_1})$ . Recall the well-known formula (see Theorem 6.3 in [3]) for the dimension of a Weyl module of a vector space  $E$ :

$$(15) \quad \dim K_\lambda E = \prod_{1 \leq i < j \leq \dim E} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Applying this to  $K_{\mu'} F$  and ignoring all multiplands that are equal to 1 we obtain precisely the formula in part a).

**Case 2:**  $f - e - r_1 + r_2 + 1 = 0$ .

We note in this case that the method in part a) works equally well but that  $\mu' = ((e-r_1)^f)$  so  $\dim K_{\mu'} F = 1$ .

**Case 3:**  $f - e - r_1 + r_2 + 1 > 0$ . Following the calculations in Proposition 3.3, we obtain  $\mu = (f^{(e-r_1)})$  and  $\sigma^\bullet((\lambda, \mu)) = ((f-r_1)^{(e-r_1)}, (e-r_1-1)^{r_1})$ . Again after some rearranging and cancellation, using formula (15) we deduce the expression in the statement of the proposition.  $\square$

As often seems the case, calculations for when  $F$  is orthogonal pose a few more difficulties than for when  $F$  is symplectic. First of all, the iterated mapping cone construction does not give us a free resolution of the proper depth if  $f = 2r_1 - r_2$  and hence we cannot thereby determine  $\text{Top}(\bar{O}_{r_1, r_2})$ .

Secondly, all the partitions  $\lambda$  appearing in the resolution (6) have even rank. Thus  $\lambda_{max} = ((r_1 - r_2 + 1)^{r_1})$  only when  $\min(r_1, r_1 - r_2 + 1)$  is even. If  $\min(r_1, r_1 - r_2 + 1)$  is odd, then

$$\lambda_{max} = ((r_1 - r_2 + 1)^{(r_1 - r_2)}, (r_1 - r_2)^{r_2}).$$

The method to calculate  $\mu$  in Proposition 3.3 still applies, but because  $\lambda_{max}$  is not a rectangle, one is led to consider more options than in Proposition 4.2. The calculations are not particularly enlightening so we do not carry them out. Nonetheless, we can conclude with a simple proposition that states which coordinate rings  $k[\bar{O}_{r_1, r_2}]$  are Gorenstein.

**Proposition 4.3.** *Let  $E$  be a vector space of dimension  $e$  and  $F$  either a symplectic or orthogonal space of dimension  $f$ .*

- a) *If  $F$  is symplectic, then  $k[\bar{O}_{r_1, r_2}]$  is Gorenstein if and only if  $f - e - r_1 + r_2 = -1$ .*
- b) *If  $F$  is orthogonal with  $f > 2r_1 - r_2$ , then  $k[\bar{O}_{r_1, r_2}]$  is Gorenstein if and only if  $f - e - r_1 + r_2 = 1$  and  $\min(r_1, r_1 - r_2 + 1)$  is even.*

*Proof.* We recall that a Noetherian ring is Gorenstein if it is Cohen-Macaulay and of type 1. (See Theorem 3.2.10 in [1].)

Part a) follows almost immediately from Proposition 4.2. Part b) of Proposition 4.2 establishes the direction ( $\Leftarrow$ ). Furthermore, we can easily rewrite the products in parts a) and c) so that all the terms in the product are greater than 1 and hence produce a type that is greater than 1. This proves the ( $\Leftarrow$ ) direction.

For part b), let us first assume that  $f > 2r_1 - r_2$  and hence that the iterated mapping cone construction provides a resolution of  $k[\bar{O}_{r_1, r_2}]$  of length equal to the codimension of  $\bar{O}_{r_1, r_2}$ . If  $\min(r_1, r_1 - r_2 + \varepsilon)$  is even then  $\lambda_{max} = (r_1 - r_2 + 1)^{r_1}$  and we can apply identical methods as Proposition 4.2 with the only change that  $\varepsilon = 1$ . On the other hand, if  $\min(r_1, r_1 - r_2 + \varepsilon)$  is odd, then since  $\lambda_{max}$  is not a rectangle, a quick check of all possibilities shows that  $\mu$  and  $\sigma^\bullet((\lambda_{max}, \mu))$  cannot both be rectangles and hence  $r(\bar{O}_{r_1, r_2}) > 1$ .  $\square$

We conclude with an example that illustrates Proposition 4.3.

**Example 4.4.** Let  $E$  be a vector space with  $\dim E = 5$  and  $F$  an orthogonal vector space with  $\dim F = 7$ . Let us choose rank conditions  $r_1 = 4$  and  $r_2 = 3$ . Since  $F$  is orthogonal,  $\varepsilon = 1$  and thus  $f - e - r_1 + r_2 - \varepsilon = 0$ . Furthermore,  $\min(r_1, r_1 - r_2 + 1) = 2$  is even and  $f > 2r_1 - r_2$  and hence  $k[\bar{O}_{r_1, r_2}]$  is Gorenstein.

Using Theorem 2.2, then decomposing Schur functors of  $F$  into irreducible  $O(F)$ -modules (see [6]), we can find the minimal resolution of  $k[\bar{O}_{r_1, r_2}]$ . It is as follows:

$$\begin{array}{c}
0 \\
\downarrow \\
K_{(3^5)}E \otimes A(-15) \\
\downarrow \\
K_{(2^5)}E \otimes V_{(1,1)}F \otimes A(-10) \oplus K_{(3,1^4)}E \otimes A(-7) \\
\downarrow \\
K_{(2^4,1)}E \otimes F \otimes A(-9) \oplus K_{(2,1^4)}E \otimes F \otimes A(-6) \\
\downarrow \\
K_{(2^4)}E \otimes A(-8) \oplus K_{(1^5)}E \otimes V_{(1,1)}F \otimes A(-5) \\
\downarrow \\
A \\
\downarrow \\
0
\end{array}$$

where  $V_{(1,1)}F$  is irreducible representation of  $\mathfrak{so}(F)$  of highest weight  $(1, 1)$ . (We used results in [6] to pass between Schur functors of  $F$  and irreducible representations of  $\mathfrak{so}(F)$ .)

As the subject of a future article, the author plans to study where these examples of Gorenstein orbit closures of codimension 4 fit into the classifications done by Kustin and Miller in [7].

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