ORTHOGONAL AND SYMPLECTIC ANALOGUES OF DETERMINANTAL IDEALS

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Abstract. We consider subvarieties of determinantal varieties determined by an additional rank equation that defines an orthogonal or symplectic structure. Such varieties simultaneously generalize usual determinantal varieties and rank varieties of symmetric or anti-symmetric matrices. In this article, we find a nontrivial class of such orthogonal or symplectic analogues of determinantal varieties for which we can provide a completely combinatorial description of the terms in a minimal resolution of the coordinate ring. The results come as an application of the geometric technique and Bott's Theorem for the cohomology of vector bundles over the Grassmannian.

1. Introduction

In [9], Kac classified all the representations of connected reductive groups with finitely many orbits. On Kac's list of representations with finitely many orbits, only a few are naturally grouped into two infinite series. The first family of representations is the type A family, namely when the representation is $V = E \otimes F$, $E = k^m$, $F = k^n$ acted on by the reductive group $G = \mathrm{SL}(E) \times \mathrm{SL}(F)$ where k is an algebraically closed field of characteristic 0. The orbit closures in this representation are the usual determinantal varieties for which the Lascoux resolution (see [11]) provides a minimal free resolution of the coordinate ring. From this minimal resolution, one can easily conclude that determinantal varieties are Cohen-Macaulay, normal and have rational singularities.

This paper studies properties for the other two-parameter families of representations with finitely many orbits. These include representations of the form $V = E \otimes F$ where $E = k^n$, F is an orthogonal (resp. symplectic) k-vector space and $G = \mathrm{SL}(E) \times \mathrm{SO}(F)$ (resp. $G = \mathrm{SL}(E) \times \mathrm{Sp}(F)$). We call these orbit closures orthogonal and symplectic analogues of determinantal varieties. The main theorem (Theorem 4.2), like Lascoux's resolution, provides a combinatorial description for the terms in a minimal free resolution of the coordinate rings for a special class of these orbits.

Consider a vector space E of dimension e and a symplectic or orthogonal vector space F of dimension f equipped with a symmetric (resp. skew-symmetric) non-degenerate bilinear product $\langle \cdot, \cdot \rangle$ for which F can have isotropic spaces of maximal dimension. The non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ defines an isomorphism $i : F \cong F^*$ by $v \mapsto \langle v, \cdot \rangle$. In the rest of this chapter, when we consider a linear map $\phi : E \to F$ we define $\phi^\star : F \to E^*$ as the composition $\phi^* \circ i$. Thus the diagram

$$E \xrightarrow{\phi} F \xrightarrow{i:\cong} F^* \xrightarrow{\phi^*} E^*$$

becomes

$$E \xrightarrow{\phi} F \xrightarrow{\phi^\star} E^* .$$

The closures of all the finite orbits in the representations we propose to study are orthogonal (resp. symplectic) analogues of determinantal varieties and can be described succinctly as

$$\bar{O}_{r_1,r_2} = \{\phi \in \mathrm{Hom}(E, F) : \mathrm{rank}\, \phi \le r_1 \text{ and } \mathrm{rank}(\phi^\star \phi) \le r_2\}$$

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where r_1 and r_2 are integers satisfying the compatibility conditions

$$0 \leq r_2 \leq r_1 \leq e \qquad \text{and} \qquad 2r_1 - r_2 \leq f \ . \tag{1}$$

The varieties $\bar{O}_{r_1,r_2}$ closely related not only to standard determinantal varieties but also to the varieties of symmetric or antisymmetric matrices with specified ranks. These varieties occur in numerous places in algebraic geometry but are a worthy object of study in their own right. In [11], Lascoux exhibits a minimal free resolution of the usual determinantal variety in terms of Weyl modules on E and F, which we denote by $K_\lambda E$ and $K_\lambda F$ where λ is a weight. However, the resolutions we obtain for $\bar{O}_{r_1,r_2}$ more closely resemble resolutions of symmetric or antisymmetric matrices with specified rank, thoroughly studied by Józefiak, Pragacz and Weyman in [8]. Therefore, for the sake of comparison, we provide a condensed description of the terms in the resolution using a different though equivalent formulation that will mesh more easily with the results of this paper.

For the variety of anti-symmetric matrices of a specified rank, we remark first that the rank must be even. The authors set $X = \mathrm{Alt}_n(k)$ as the affine set of anti-symmetric $n \times n$-matrices over a field k and Y_{2p} the variety in X of matrices of rank at most $2p$. Call $E = k^n$ and, for any partition $\lambda = (a_1, \ldots, a_t | b_1, \ldots, b_t)$ written in Frobenius notation, define $d_s(\lambda) = \sum_{i=1}^{t} a_i$. Then the i'th component of a minimal free resolution of $k[Y_{2p}]$ over $A = k[X]$ is equal to

$$\bigoplus_{d_s(\lambda)=i} K_\lambda E \otimes A\Big(- \frac{|\lambda|}{2} \Big) \tag{2}$$

where summation is only taken over partitions $\lambda = (a_1, \ldots, a_t | b_1, \ldots, b_t)$ that satisfy $b_i = a_i + 2p + 1$.

For symmetric matrices, in [8] the authors set $X = \mathrm{Sym}_n(k)$ the affine set of symmetric $n \times n$-matrices over k and Y_r as the variety of all matrices of rank at most r. Again call $E = k^n$ and, for any partition $\lambda = (a_1, \ldots, a_t | b_1, \ldots, b_t)$, define $d_o(\lambda) = \sum_{i=1}^{t} (a_i - 1)$. The authors then construct a minimal free resolution of $k[Y_r]$ over $A = k[X]$ whose i'th component is equal to

$$\bigoplus_{d_o(\lambda)=i} K_\lambda E \otimes A\Big(- \frac{|\lambda|}{2} \Big) \tag{3}$$

where summation is only taken over partitions $\lambda = (a_1, \ldots, a_t | b_1, \ldots, b_t)$ that satisfy $b_i = a_i + r - 1$ and t is even.

In all three of the above cases, the authors use the minimal resolutions they construct to determine that the varieties are Cohen-Macaulay, are normal and have rational singularities.

The goal of this article is to apply similar techniques as the above authors to study the varieties $\bar{O}_{r_1,r_2}$ and obtain (at least for some of these varieties) a resolution of the coordinate ring $k[\bar{O}_{r_1,r_2}]$ as an A-module where $A = k[X]$ and $X = \mathrm{Hom}(E, F)$. This article is organized as follows.

In section 2, we first construct a desingularization for the variety $\bar{O}_{r_1,r_2}$. Then we briefly state what is generally called the "geometric technique" - a collection of theorems that allows one to calculate the complex $F_\bullet$ which, under certain conditions, is a minimal resolution of the coordinate ring of certain affine varieties. The proofs behind the geometric technique would take us far afield of our goal so we only provide a reference for them. We apply the geometric technique to the varieties $\bar{O}_{r_1,r_2}$ and restrict our attention to what we call *special orbits*, varieties for which either $2r_1 - r_2 = f$ or $r_1 = e$ and r_2 has the same parity of f. These orbit closures enjoy particularly simple desingularizations which then lead to tractable calculations.

In section 3, we study the combinatorial problems in the Weyl groups of $\mathrm{Sp}(F)$ and $\mathrm{O}(F)$ that allow one to calculate the cohomology groups required by the geometric technique. We recommend that the reader skip this section for a first reading of this article.

Finally, in section 4 we establish the main theorem (Theorem 4.2) of this paper which calculates the $F_\bullet$-complex for closures of special orbits $\bar{O}_{r_1,r_2}$. The theorems of the geometric technique then allow us to conclude the following.

- If F is symplectic, then for special orbits the $F_\bullet$-complex provides a minimal free resolution of $k[\bar{O}_{r_1,r_2}]$ as a $k[\mathrm{Hom}(E,F)]$-module. One immediately deduces that $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay and normal. Furthermore, in the case where $\dim F = 2r_1 - r_2$, the variety $\bar{O}_{r_1,r_2}$ has rational singularities. In section 4.3, we use Theorem 4.2 to calculate examples of minimal resolutions to $\bar{O}_{r_1,r_2}$.
- If F is orthogonal then three situations occur:
  a) if $\dim F = 2r_1 - r_2$ and $r_2 \neq 0$, then we can only conclude that the normalization of $\bar{O}_{r_1,r_2}$ has rational singularities.
  b) if $\dim E = r_1$ and $2r_1 - r_2 < dimF$ then $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay and normal.
  c) if $f = \dim F$ is even, $r_1 = \frac{f}{2}$ and $r_2 = 0$, then the $F_\bullet$-complex provides a minimal free resolution of $k[\bar{O}_{r_1,r_2}]$ as a $k[\mathrm{Hom}(E,F)]$-module and the variety $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay, normal and has rational singularities.

The perceived inability to prove Cohen-Macaulayness or normality for general orbits using the geometric technique is not due entirely to insufficent methods since in a future paper the author presents examples of varieties of the form $\bar{O}_{r_1,r_2}$ that are not Cohen-Macaulay.

With the assistance of a computer algebra program that incorporates Bott's Theorem, the geometric technique and the universal character theorem that relates characters in different classical groups (see Koike and Terada [10]), we can calculate resolutions for all orbits $\bar{O}_{r_1,r_2}$ including non-special ones. However, the combinatorics involved are rather intractable and do not readily lend themselves to generalizations as we obtained for special orbit sin this paper. Nonetheless, based on observations from these calculations, we conjecture that all orbits $\bar{O}_{r_1,r_2}$ are Cohen-Macaulay, normal and have rational singularities *unless* F is orthogonal and $\dim F = 2r_1 - r_2$ in which case we only know that its normalization has rational singularities.

2. The Strategy of the Geometric Technique

The strategy we use in this chapter, involves presenting desingularizations for $\bar{O}_{r_1,r_2}$ and applying geometric theorems that provide free resolutions (sometimes minimal) for the coordinate ring of this variety. These theorems then allow us to answer questions about normality and rationality of singularities.

2.1. Orbit Closures and Desingularizations.

Definition 2.1. Let V be a vector space with a nondegenerate pairing $\langle\cdot,\cdot\rangle$ between V and V^* (not necessarily the canonical one). Let S be a subspace of a vector space V. We define the association $\check{}$ on the set of subspaces of V to the set of subspaces of V^* by $\check{S} = \{\lambda \in V^* : \langle v, \lambda\rangle = 0 \text{ for all } w \in S\}$.

In order to view the association $\check{}$ as a functor, we must be clear on the categories in which this association $\check{}$ operates. Let **SubVs** be the category of subspaces of vector spaces. That is to say that the objects of our category are pairs of vector spaces (V,W) with $W \subset V$ and an arrow from (V_1,W_1) to (V_2,W_2) is a linear map $f : V_1 \longrightarrow V_2$ such that $f(W_1) \subset W_2$.

Lemma 2.2. *The association* $\check{}$ *is a contravariant functor* **SubVs**$\rightarrow$**SubVs** *given by*

$$(V,W) \mapsto (V^*, \check{W}) .$$

Proof. By abuse of notation, we will usually refer to the pair (V, W) as just W, where W is a subspace of V and we assume that V is understood by context. The association $\check{}$ maps (V_1, W_1) to $(V_1^*, \check{W}_1)$ and it acts on the arrows simply by $\check{f} = f^\star$, the adjoint linear transformation:

$$f : (V_1, W_1) \longrightarrow (V_2, W_2)$$
$$f^\star : (V_2^*, \check{W}_2) \longrightarrow (V_1^*, \check{W}_1)$$

This action is well-defined on morphisms of **SubVs** since for all $\lambda \in V_2^*$ and $v \in V_1$, $\langle v, f^\star(\lambda)\rangle_1 = \langle f(v), \lambda\rangle_2$. Hence, if $\lambda \in \check{W}_2$, then $f^\star(\lambda) \in \check{W}_1$ since $f(v) \in W_2$.

Contravariance follows immediately from the contravariance of the adjoint linear transformation. □

Remark 2.1. Note that if for $v \in V$ and $\lambda \in V^*$ the pairing is given by $\langle v, \lambda\rangle = \lambda(v)$, then the adjoint morphism is simply the dual map.

We now return to the orbit closure $\bar{O}_{r_1,r_2}$ and provide a rather simple desingularization. Consider the following product of a Grassmannian with an isotropic Grassmannian, $\mathcal{F}_{r_1,r_2} = \mathbb{G}(e - r_2, E) \times \mathbb{IG}(r_1 - r_2, F)$. The space $\mathcal{F}_{r_1,r_2}$ is a product of homogeneous spaces and by that virtue is a projective variety. Furthermore, consider the variety

$$X_{r_1,r_2} = \{((R, S), \phi) \in \mathcal{F}_{r_1,r_2} \times \mathrm{Hom}(E, F) : \phi(E) \subset \check{S}, \phi(R) \subset S\} \,. \tag{4}$$

Clearly X_{r_1,r_2} is the total space of a vector subbundle over $\mathcal{F}_{r_1,r_2}$ of the trivial bundle $\mathcal{F}_{r_1,r_2} \times \mathrm{Hom}(E, F)$. We call p and q the first and second projections from $\mathcal{F}_{r_1,r_2} \times \mathrm{Hom}(E, F)$ and π the restriction of q to X_{r_1,r_2}. We remark right away that the variety X_{r_1,r_2} is nonsingular.

Proposition 2.3. *The map* $\pi : X_{r_1,r_2} \longrightarrow \mathrm{Hom}(E, F)$ *is a desingularization of* $\bar{O}_{r_1,r_2}$.

Proof. Consider the open subset U in $\mathrm{Hom}(E, F)$.

$$U = \{\phi \in \mathrm{Hom}(E, F) \bigm| \mathrm{rank}\,\phi = r_1, \quad \mathrm{rank}\,\phi^\star\phi = r_2\} \,.$$

Let $((R, S), \phi) \in \pi^{-1}(U)$. From the condition that $\phi(E) \subset \check{S}$ and Lemma 2.2, we obtain $\phi^\star(S) \subset 0$ since $\check{E} = 0$. Thus $S \subset \ker\phi^\star$. Since $\phi(R) \subset S$ we also obtain $R \subset \ker(\phi^\star\phi)$. However, since $\dim R = e - r_2$ and $\mathrm{rank}\,\phi^\star\phi = r_2$, then $\dim R = \dim\ker(\phi^\star\phi)$ and hence $R = \ker(\phi^\star\phi)$.

Furthermore, the other condition $\phi(R) \subset S$ implies $\phi^\star(\check{S}) \subset \check{R}$ and hence that $\mathrm{Im}(\phi^\star\phi) \subset \check{R}$. However, $\dim\check{R} = e - (e - r_2) = r_2 = \mathrm{rank}\,\phi^\star\phi$ and hence $\check{R} = \mathrm{Im}(\phi^\star\phi)$. Therefore $\check{S} \subset (\phi^\star)^{-1}(\mathrm{Im}\ \phi^\star\phi)$ and since these two subspaces of F again have the same dimension, they are equal.

Therefore, on U, the inverse map π^{-1} is well-defined and is the algebraic map

$$\phi \mapsto ((\ker(\phi^\star\phi), \big((\phi^\star)^{-1}(\mathrm{Im}\ \phi^\star\phi)\big)\check{}\ ), \phi).$$

Hence $\pi|_{\pi^{-1}(U)}$ is an isomorphism and π is a birational isomorphism. □

This desingularization allows us to determine the codimension of $\bar{O}_{r_1,r_2}$.

Corollary 2.4. *The codimension of the orbit closure* $\bar{O}_{r_1,r_2}$ *in* $\mathrm{Hom}(E, F)$ *is*

$$\mathrm{codim}\,\bar{O}_{r_1,r_2} = (e - r_1)(f - r_1) + \frac{(r_1 - r_2)(r_1 - r_2 + \varepsilon)}{2} \tag{5}$$

where

$$\varepsilon = \begin{cases} 1 & \textit{if } F \textit{ is orthogonal} \\ -1 & \textit{if } F \textit{ is symplectic} \end{cases}$$

Proof. Since X_{r_1,r_2} is a desingularization of $\bar{O}_{r_1,r_2}$ we know that $\dim X_{r_1,r_2} = \dim \bar{O}_{r_1,r_2}$. With ε as defined above, we have

$$\dim \mathcal{F}_{r_1,r_2} \times \operatorname{Hom}(E,F) = r_2(e-r_2) + (f-r_1+r_2)(r_1-r_2) - \frac{(r_1-r_2+\varepsilon)(r_1-r_2)}{2} + ef$$

Therefore, we find that

$$\dim X_{r_1,r_2} = \dim \mathcal{F}_{r_1,r_2} \times \operatorname{Hom}(E,F) - \Big(ef - (e-r_2)(r_1-r_2) - (f-r_1+r_2)r_2\Big)$$

After a simple calculation, we get

$$\operatorname{codim} \bar{O}_{r_1,r_2} = ef - \dim X_{r_1,r_2} = (e-r_1)(f-r_1) + \frac{(r_1-r_2)(r_1-r_2+\varepsilon)}{2}$$

□

2.2. **The geometric technique.** We can now begin to fit our situation into methods of using geometric techniques to calculate syzygies (Chapter 5 of [14], *Cohomology of Vector Bundles and Syzygies*). For completeness, we restate the set-up and the basic theorems we employ, and refer the reader to [14] for proofs.

We fix an algebraically closed field k of any characteristic and consider a projective variety V over k. Let $\mathbb{A}^N_k$ be the affine space and we view $V \times \mathbb{A}^N_k$ as a trivial vector bundle of dimension N over V. Let us consider the subvariety Z which is the total space of a subbundle $\mathcal{T}$ of $V \times \mathbb{A}^N_k$. Let p and q be the first and second natural projections from $V \times \mathbb{A}^N_k$ and let q' be the restriction of q to Z. Let $Y = q(Z)$. We have the diagram:

$$\begin{array}{ccc} Z & \hookrightarrow & V \times \mathbb{A}^N_k \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \hookrightarrow & \mathbb{A}^N_k \end{array}$$

We then have an exact sequence of vector bundles over V:

$$0 \longrightarrow \mathcal{T} \longrightarrow V \times \mathbb{A}^N_k \longrightarrow \mathcal{T}' \longrightarrow 0$$

We denote by A the coordinate ring of $\mathbb{A}^N_k$ which is $k[x_1, \ldots, x_N]$. Furthermore, we identify the sheaves on $\mathbb{A}^N_k$ with A-modules. With these notations in place, we are in a position to state the three main theorems we will use.

Proposition 2.5. *Proposition 5.1.1 in* [14].

a) *Let $\xi = (\mathcal{T}')^*$ and $t = \operatorname{rank} \xi$. The locally free resolution of the sheaf $\mathcal{O}_Z$ as an $\mathcal{O}_{V\times\mathbb{A}^N}$-module is given by the Koszul complex*

$$\mathcal{K}_\bullet(p^*\xi) : 0 \to \bigwedge^t(p^*\xi) \to \ldots \to \bigwedge^2(p^*\xi) \to p^*\xi \to \mathcal{O}_{V\times\mathbb{A}^N} \ .$$

The differentials in this complex are homogeneous of degree 1 in the coordinate functions on X.

b) *The direct image $p_*(\mathcal{O}_Z)$ can be identified with the sheaf of algebras $Sym(\eta)$ where $\eta = Z^*$*

Theorem 2.6. *Basic Theorem 5.1.2 in* [14]. *Let $\mathcal{V}$ be a vector bundle over V and consider the $\mathcal{O}_{V\times\mathbb{A}^N}$-module $M(\mathcal{V}) = \mathcal{O}_Z \otimes p^*\mathcal{V}$. We define the following collection of free graded A-modules:*

$$F_i(\mathcal{V}) = \bigoplus_{j\geq 0} H^j(V, \bigwedge^{i+j} \xi \otimes \mathcal{V}) \otimes_k A(-i-j) \ .$$

a) *There exist minimal differentials*

$$d_i(\mathcal{V}) : F_i(\mathcal{V}) \longrightarrow F_{i-1}(\mathcal{V})$$

of degree 0 such that $F_\bullet(\mathcal{V})$ is a complex of graded free A-modules with

$$H_{-i}(F_\bullet(\mathcal{V})) = \mathcal{R}^i q_* M(\mathcal{V}) \ .$$

In particular, the complex $F_\bullet(\mathcal{V})$ is exact in positive degrees.

b) *The sheaf $\mathcal{R}^i q_* M(\mathcal{V})$ is equal to $H^i(Z, M(\mathcal{V}))$ and it can also be identified with the graded A-module $H^i(V, Sym(\eta) \otimes \mathcal{V})$.*

c) *If $\phi : M(\mathcal{V}) \to M(\mathcal{V}')(n)$ is a morphism of graded sheaves, then there exists a morphism of complexes $F_\bullet(\phi) : F_\bullet(\mathcal{V}) \to F_\bullet(\mathcal{V}')(n)$. Its induced map $H_{-i}(f_\bullet(\phi))$ can be identified with the induced map*

$$H^i(Z, M(\mathcal{V})) \longrightarrow H^i(Z, M(\mathcal{V}'))(n) \ .$$

When $\mathcal{V}$ is a trivial one-dimensional bundle on V, then we'll denote the complex $F_\bullet(\mathcal{V})$ simply by $F_\bullet$.

Finally, the next theorem is the tool we will use to answer geometric questions as it delineates when $F_\bullet$ is the free resolution of the coordinate ring of the variety Y.

Theorem 2.7. *Theorem 5.1.3 in* [14]. *Let us assume that the map $q' : Z \to Y$ is a birational isomorphism. Then the following properties hold:*

a) *The module $q'_* \mathcal{O}_Z$ is the normalization of $k[Y]$.*

b) *If $\mathcal{R}^i q'_* \mathcal{O}_Z = 0$ for $i > 0$, then $F_\bullet$ is a finite free resolution of the normalization of $k[Y]$ treated as an A-module.*

c) *If $\mathcal{R}^i q'_* \mathcal{O}_Z = 0$ for $i > 0$ and $F_0 = H^0(V, \bigwedge^0 \xi) \otimes A = A$, then Y is normal and it has rational singularities.*

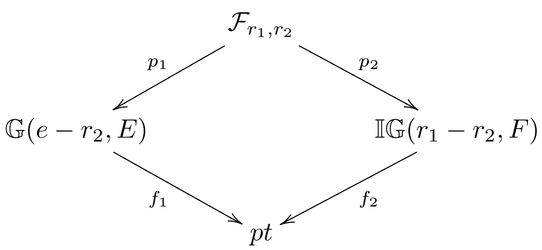
2.3. **Application of the Geometric Technique to $\bar{O}_{r_1,r_2}$.** We now return to the study of the orbit closure $\bar{O}_{r_1,r_2}$. We can apply the geometric technique by taking $\mathbb{A}^N = \mathrm{Hom}(E,F)$, $Y = \bar{O}_{r_1,r_2}$, $V = \mathcal{F}_{r_1,r_2} = \mathbb{G}(e-r_2, E) \times \mathbb{IG}(r_1 - r_2, F)$ and $Z = X_{r_1,r_2}$. Notice also that q' corresponds to the desingularization π.

In the language of vector bundles over $\mathcal{F}_{r_1,r_2}$, $\mathcal{F}_{r_1,r_2} \times \mathrm{Hom}(E,F)$ is the total space of a trivial bundle $\mathcal{E}$ and the variety $Z = X_{r_1,r_2}$ is the total space of a bundle $\mathcal{T}$. Furthermore, $\mathcal{T}$ is a subbundle of $\mathcal{E}$ and we have the exact sequence:

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{E} \longrightarrow \mathcal{T}' \longrightarrow 0 \tag{6}$$

By Proposition 2.5, setting $\xi = (\mathcal{T}')^*$ is the dual bundle to $\mathcal{T}'$, the Koszul complex of the sheaf $p^*\xi$ provides a locally free resolution of $\mathcal{O}_Z$ as an ideal sheaf of $\mathcal{O}_{\mathcal{F}_{r_1,r_2} \times \mathrm{Hom}(E,F)}$.

We cannot obtain ξ explicitly as a direct sum of tautological vector bundles but we can describe it as a filtration of vector bundles. The projective variety $\mathcal{F}_{r_1,r_2}$ comes with projection morphisms:



Let $\mathcal{R}$ and $\mathcal{S}$ be the tautological vector bundles over $\mathbb{G}(e-r_2, E)$ and $\mathbb{IG}(r_1-r_2, F)$ respectively. We will also denote by $\mathcal{R}$ and $\mathcal{S}$ the pull-backs of these bundles to $\mathcal{F}_{r_1,r_2}$, namely $p_1^*\mathcal{R}$ and $p_2^*\mathcal{S}$.

Proposition 2.8. *The following sequence of vector bundles over $\mathcal{F}_{r_1,r_2}$ is exact:*

$$0 \to E \otimes \mathcal{S} \to \xi \to \mathcal{R} \otimes \check{\mathcal{S}}/\mathcal{S} \to 0 \tag{7}$$

where $\check{\mathcal{S}}$ is defined in Lemma 2.2.

Proof. We work with the following flags of the trivial vector bundles E and F: $E \supset \mathcal{R} \supset 0$ and $F \supset \check{\mathcal{S}} \supset \mathcal{S} \supset 0$. The conditions defining $Z = X_{r_1,r_2}$ describe $\mathcal{T}$ as in the composition series

$$0 \longrightarrow E^* \otimes \mathcal{S} \longrightarrow \mathcal{T} \longrightarrow E/\mathcal{R} \otimes \check{\mathcal{S}}/\mathcal{S} \longrightarrow 0\ .$$

Hence for $\mathcal{T}'$ and ξ we get short exact sequences:

$$0 \longrightarrow \mathcal{R}^* \otimes \check{\mathcal{S}}/\mathcal{S} \longrightarrow \mathcal{T}' \longrightarrow E^* \otimes F/\check{\mathcal{S}} \longrightarrow 0$$
$$0 \longrightarrow E \otimes (F/\check{\mathcal{S}})^* \longrightarrow \xi \longrightarrow \mathcal{R} \otimes (\check{\mathcal{S}}/\mathcal{S})^* \longrightarrow 0\ .$$

However, by construction of $\check{\mathcal{S}}$, we notice that the following isomorphisms are canonical $F/\check{\mathcal{S}} \cong \mathcal{S}^*$ and $(\check{\mathcal{S}}/\mathcal{S})^* \cong \check{\mathcal{S}}/\mathcal{S}$. The proposition follows. □

As a consequence of Proposition 2.8, to calculate the Koszul complex $K_\bullet(p^*\xi)$, we must use this composition of ξ to calculate the terms $\bigwedge^i \xi$. In general, all we can say is that $\bigwedge^i \xi$ has a filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_i \subset \mathcal{S}_{i+1} = \bigwedge^i \xi$$

with

$$\mathcal{S}_{j+1}/\mathcal{S}_j \cong \bigwedge^j (E \otimes \mathcal{S}) \otimes \bigwedge^{i-j} (\mathcal{R} \otimes \check{\mathcal{S}}/\mathcal{S})\ .$$

Combining Künneth theorems and Bott's Theorem, one can explicitly calculate the cohomology groups for each of the terms in the above composition series. This composition series of $\bigwedge^i \xi$ then defines a spectral sequence that converges to $H^*(V, \bigwedge^i \xi)$. In principle therefore, we can calculate all the necessary cohomology groups. However, in general such calculations are quite intractable so we restrict ourselves to situations where one does not require spectral sequences.

2.4. **Special Orbits of $\bar{O}_{r_1,r_2}$.** We point out that the desingularization that we provided in Proposition 2.3 is only one of many one could study. Furthermore, for some particular values of r_1 and r_2, we can provide a desingularization of orbit closures $\bar{O}_{r_1,r_2}$ that lead to a particularly simple expression for the bundle ξ.

Every orbit closure $\bar{O}_{r_1,r_2}$ depends only on four parameters $\dim E = e$, $\dim F = f$, and the ranks r_1 and r_2. Equation 1, lists the compatibility conditions that must hold between these parameters. Whenever one of the inequalities degenerates to an equality, then the orbit only depends on one parameter besides e and f, thereby simplifying the problem.

a) If $r_2 = 0$, the orbit closure $\bar{O}_{r_1,0}$ is the variety of maps $\phi \in \mathrm{Hom}(E, F)$ such that $\mathrm{rank}\,\phi \le r_1$ and Im ϕ is in an isotropic subspace of F.
b) If $r_1 = r_2$, maps ϕ in the orbit closure $\bar{O}_{r_1,r_1}$ have the properties that $\langle\cdot,\cdot\rangle|_{\mathrm{Im}\,\phi}$ is nondegenerate and $f = \phi^\star\phi : E \to E^*$ is symmetric or skew-symmetric depending on whether $\langle\cdot,\cdot\rangle$ is.
c) If $r_1 = e$ then there is a surjective morphism

$$\begin{aligned} j : \bar{O}_{e,r_2} &\longrightarrow Y_{r_2} \\ \phi\ &\longmapsto \phi^\star\phi \end{aligned}$$

where Y_{r_2} is the variety of skew-symmetric $e \times e$-matrices or rank less than or equal to r_2 in the symplectic case, and is the variety of symmetric $e \times e$-matrices of rank less than or equal to r_2 in the orthogonal case.

d) The equality $2r_1 - r_2 = f$ is a less obvious condition. We will wait until after Proposition 2.9 to discuss the simplification that follows from this equality.

We will focus in particular on cases c) and d) in the above list. In doing so, we make use of the variety defined in the following proposition.

Proposition 2.9. *Let E be a vector space of dimension e, F a symplectic or orthogonal vector space of dimension f and define the variety X'_s as follows:*

$$X'_s = \{(S, \phi) \in \mathbb{IG}(s, F) \times \operatorname{Hom}(E, F) \,:\, \phi(E) \subset \check{S}\}$$

Depending on the dimensions of E, F and S, three cases occur:

a) *If $e \geq f - s$ then the projection onto the second factor $\pi' : X'_s \longrightarrow \operatorname{Hom}(E, F)$ is a desingularization of $\bar{O}_{f-s,f-2s}$.*

b) *If $f - 2s < e < f - s$ then projection onto the second factor $\pi' : X'_s \longrightarrow \operatorname{Hom}(E, F)$ surjects onto $\bar{O}_{e,f-2s}$ and generically each fiber is a grassmannian of maximal isotropic subspaces in an orthogonal or symplectic (the same as F) vector space of dimension $2(f - e - s)$.*

c) *If $e \leq f - 2s$ then projection onto the second factor $\pi' : X'_s \longrightarrow \operatorname{Hom}(E, F)$ surjects onto $\bar{O}_{e,e} = \operatorname{Hom}(E, F)$.*

Proof. We consider and work over the open subset U' in X'_s such that $\phi(E)$ has maximal dimension, either $r_1 = f - s$ if $e \leq f - s$ or $r_1 = e$ otherwise. Note also that in all cases, the rank condition $r_2 = \operatorname{rank} \phi^\star\phi$ corresponds to the rank of $\langle \cdot, \cdot \rangle|_{\operatorname{Im}\phi}$. This is because for any two vectors $v, w \in E$, $\langle \phi(v), \phi(w) \rangle = \langle v, \phi^\star\phi(w) \rangle$ and $\langle \, , \, \rangle$ is non-degenerate on $E \times E^*$.

If $e \geq f - s$, then the condition $\phi(E) \subset \check{S}$ means that $\phi(E) = \check{S}$ and hence that $S = (\operatorname{Im}\phi)\check{}$. Furthermore, since $S \subset \operatorname{Im}\phi$ is an isotropic subspace of F, the rank of $\langle \cdot, \cdot \rangle|_{\operatorname{Im}\phi}$ is equal to $f - 2s$. Thus the map $\phi \mapsto (\phi, (\operatorname{Im}\phi)\check{})$ defines the inverse of $\pi' : X'_s \longrightarrow \operatorname{Hom}(E, F)$ over the orbit closure $\bar{O}_{f-s,f-2s}$ making π' a birational isomorphism. This proves a).

If $f - 2s < e < f - s$ then for all pairs (ϕ, S) in U', $\operatorname{rank}\phi = e$ and $\operatorname{Im}\phi \subsetneq S$. For any subspace W of the symplectic (resp. orthogonal) space F, define $\tilde{W}$ as the maximal subspace $V \subset \check{W}$ such $(V, \langle \cdot, \cdot \rangle|_V)$ is non-degenerate. This subspace is well-defined by the identity $\check{W} = (W \cap \check{W}) \oplus \tilde{W}$. Now it is not difficult to see that the condition $\widetilde{\phi(E)} \subset \check{S}$ implies that $\check{S}$ is equal to the direct sum of $\operatorname{Im}\phi$ and a maximal isotropic subspace H of $\widetilde{\operatorname{Im}\phi}$. Consequently, $S = (\operatorname{Im}\phi \cap (\operatorname{Im}\phi)\check{}) \oplus H$.

Calculating dimensions of relevant spaces, we get $\dim \operatorname{Im}\phi \cap (\operatorname{Im}\phi)\check{} = e - r_2 = e - f + 2s$. Thus $\dim \widetilde{\operatorname{Im}\phi} = f - e - (e - f + 2s) = 2(f - e - s)$. Part b) follows.

If $e < f - 2s$ then $\langle \cdot, \cdot \rangle|_{\operatorname{Im}\phi}$ is non-degenerate so that $r_1 = r_2 = \dim E$. Hence the orbit closure $\bar{O}_{e,e} = \operatorname{Hom}(E, F)$. □

Let us look at a few consequences of this proposition. First, let us suppose that the ranks r_1 and r_2 satisfy the equality in the necessary compatibility condition $2r_1 - r_2 \leq f$, i.e. $2r_1 - r_2 = f$. In this case, we can set $s = f - r_1$ which implies that $s = \frac{f - r_2}{2}$. Then $r_1 = f - s \leq e$ so by part a) of Proposition 2.9, π' is a desingularization of $\bar{O}_{r_1,r_2}$.

Consider next the case when $r_1 = e$. Again we set $s = \frac{f - r_2}{2}$. Since

$$2r_1 - r_2 = 2e - r_2 \leq f$$

then $e \leq f - s$ and at the same time, $f - 2s \leq e$. If $e = r_2 = f - 2s$, we are in case c) of the above proposition. On the other hand, if $r_2 < e \leq f - s$, then we are in case b). Let us assume the latter and call

$$U = \{\phi \in \operatorname{Hom}(E, F) \,:\, \operatorname{rank}\phi = e, \operatorname{rank}\phi^\star\phi = r_2\}$$

the open subset in $\mathrm{Hom}(E,F)$. Then for any point $x \in U$, the fiber $\pi'^{-1}(x)$ is an isotropic Grassmannian of maximal rank in an orthogonal (resp. symplectic) vector space of dimension $2(f-e-s)$. In particular, when F is orthogonal $\pi'^{-1}(x)$ consists of two connected components while if F is symplectic $\pi'^{-1}(x)$ is connected.

Because of these remarks, we distinguish a special class of orbits:

Definition 2.10. Consider the orbit O_{r_1,r_2}. If $2r_1 - r_2 = f$ or if $r_1 = e$ and r_2 has the same parity as f then we call O_{r_1,r_2} a **special orbit**.

Now, for convenience, we summarize what we need to know to utilize the geometric technique on special orbits.

Proposition 2.11. *Let $Y = \bar{O}_{r_1,r_2}$ be the closure of a special orbit, that is to say the parameters satisfy $e = r_1$ or $f = 2r_1 - r_2$. Then:*

a) *Let $s \le \frac{f}{2}$ and $\mathcal{S}$ be the tautological bundle over $\mathbb{IG}(s,F)$. Then the variety X'_s is the total space of a vector bundle $\mathcal{T}$ over the isotropic grassmannian $\mathbb{IG}(s,F)$ and $\xi = (\mathcal{T}')^* \cong E \otimes \mathcal{S}$.*
b) *If $2r_1 - r_2 = f$, then setting $s = \frac{f-r_2}{2}$ one has a morphism $q' : X'_s \to Y$ that is a birational isomorphism and $q'_*(\mathcal{O}_{X'_s}) = \mathcal{O}_{\tilde{Y}}$, where $\tilde{Y}$ is the normalization of the scheme Y.*
c) *If $r_1 = e$ but $2r_1 - r_2 < f$ then with $s = \frac{f-r_2}{2}$, the morphism $q' : X'_s \to Y$ is a surjection and $q'_*(\mathcal{O}_{X'_s}) = H^0(X'_s, \mathcal{O}_{X'_s})\tilde{}$, the sheaf associated to the module $H^0(X'_s, \mathcal{O}_{X'_s})$.*

Proof. Part a) follows from the definition of X'_s in 2.9 and the fact that as bundles $(F/\check{\mathcal{S}})^* \cong S$.

Part b) follows from Theorem 2.7 in our discussion about the geometric technique applied to Proposition 2.9.

Part c) is nearly trivial. It follows from part b) of 2.9 and Proposition III.8.5 in [7]. □

Consequently, in the case of special orbits, since ξ has such a simple expression, we can calculate the cohomology groups $H^j(\mathbb{IG}(s,F), \bigwedge^i \xi)$ and hence the terms of the $F_\bullet$ complex without relying on spectral sequence calculations. Since $\xi = E \otimes \mathcal{S}$ is a subbundle of $E \otimes F^*$, in concrete terms, we mean that $\xi = E \otimes (F/\check{\mathcal{S}})^*$. With this in mind, we recall that the Cauchy Formula gives:

$$\bigwedge^i \xi \cong \bigoplus_{\lambda:|\lambda|=i} K_{\lambda'}E \otimes K_\lambda (F/\check{\mathcal{S}})^* \cong \bigoplus_{\lambda:|\lambda|=i} K_{\lambda'}E \otimes K_\beta \mathcal{S}$$

where

$$\beta = (-\lambda_s, \ldots, -\lambda_2, -\lambda_1)$$

and λ' is the conjugate partition to λ. All non-zero terms arise from partitions λ in the rectangle with s rows and $\dim E = m$ columns.

Künneth theorems allow us to simplify the cohomology calculations since with char $k = 0$ we have:

$$H^j(\mathbb{IG}(s,F), K_{\lambda'}E \otimes K_\beta \mathcal{S}) = K_{\lambda'}E \otimes H^j(\mathbb{IG}(s,F), K_\beta \mathcal{S}) \;. \tag{8}$$

Finally, by Theorem 2.6 the terms in the $F_\bullet$ complex we with to calculate are given by

$$F_i = \bigoplus_{j \ge 0} \bigoplus_{\lambda:|\lambda|=i+j} K_{\lambda'}E \otimes H^j(\mathbb{IG}(s,F), K_\beta \mathcal{S}) \qquad \text{where } \beta = (-\lambda_s, \ldots, -\lambda_1) \;. \tag{9}$$

Calculating the cohomology groups $H^j(\mathbb{IG}(s,F), K_\beta \mathcal{S})$ requires the use of Bott's Theorem. Even though Bott's Theorem and its variants are essential to our calculations, we refer the reader to Chapter 4 of [14] for a complete treatment as not to take too wide of a detour in this article. However, we mention that if F is an orthogonal (resp. symplectic) space, then $H^j(\mathbb{IG}(s,F), K_\beta \mathcal{S})$ is either 0 or $V_\mu F$, the irreducible representation of $\mathrm{O}(E)$ (resp. $\mathrm{Sp}(F)$) of some highest weight μ.

3. Combinatorics of the Weyl group

Utilizing Bott's Theorem boils down to finding all partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ that lead to a weight $\beta = (-\lambda_s, \dots, -\lambda_1, 0, \dots, 0) \in \mathbb{Z}^u$ such that there exists $\sigma \in \mathcal{W}$ with $\sigma^\bullet(\beta)$ a dominant weight.

In this section, we study the combinatorics related to this problem for the three relevant cases, namely when F is symplectic, F is odd orthogonal and F is even orthogonal. We encourage the reader to skip this section and return only for the details that undergird the algebraic and geometric results which we consolidate in section 4.

3.1. **Some combinatorics of the Weyl group of** $\mathrm{Sp}(F)$**.** In this subsection, we introduce a few combinatorial notions related to the dotted action of the Weyl group $\mathcal{W}_u$ of $\mathrm{Sp}(F)$. We ask the reader to bear with us as we present these notions for from them will ensue a succint description the resolution of the coordinate ring of $\bar{O}_{r_1,r_2}$ at least when $\bar{O}_{r_1,r_2}$ is a special orbit.

If $\dim F = 2u$, the Weyl group for $\mathrm{Sp}(F)$ is given by $\mathcal{W}_u = \mathbb{Z}_2^u \rtimes \mathcal{S}_u$ and is generated the usual generators of $\mathcal{S}_u$, the symmetric group on u elements, and the element σ_0 that acts on the element $(t_1, t_2, \dots, t_u)$ in the weight space as follows:

$$\sigma_0(t_1, t_2, \dots, t_{u-1}, t_u) = (t_1, t_2, \dots, t_{u-1}, -t_u) .$$

There are two useful ways to describe an element of the Weyl group. First, we may write an element $\sigma \in \mathcal{W}_u$ as a products of cycles in $\mathcal{S}_u$ and σ_0. For example, one such Weyl group element which will recur later on is $\sigma_k = (k, k+1, \dots, u)\sigma_0(s, s+1, \dots, u)^{-1}$ where s is an integer less than u, fixed by context. Note that σ_k acts on weights as follows:

$$\sigma_k(t_1, t_2, \dots, t_u) = (t_1, \dots, t_{k-1}, -t_s, t_k, \dots, t_{s-1}, \widehat{t_s}, t_{s+1}, \dots, t_u) .$$

The second and more common way of writing Weyl group elements in the symplectic case is to use signed permutations. In this language, for all $i \in \{1, \dots, u\}$, $\sigma(u) \in \{-u, \dots, -1, 1, \dots, u\}$ and that $|\sigma|$ is a standard permutation of $\mathcal{S}_u$. Then σ acts on the weights of F as:

$$\sigma(t_1, \dots, t_u) = (\mathrm{sign}(\sigma(1))t_{\sigma(1)}, \mathrm{sign}(\sigma(2))t_{\sigma(2)}, \dots, \mathrm{sign}(\sigma(u))t_{\sigma(u)}) .$$

Note that written as a signed permutation, $\sigma_k = (1, \dots, k-1, \hat{k}, k+1, \dots, s, -k, s+1, \dots, u)$.

In the following examples, $\rho = (u, u-1, \dots, 2, 1)$ is the half sum of the positive roots and for any σ in $\mathcal{W}_u$ we define the dotted action of Weyl elements on symplectic weights by $\sigma^\bullet(\beta) = \sigma(\rho+\beta) - \rho$. As we will use Bott's Theorem to calculate the cohomolgy of $\bigwedge^i \xi$, we will be interested in knowing which partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ produce a non-zero cohomology, i.e. which weights $\beta = (-\lambda_s, \dots, -\lambda_1, 0, \dots, 0)$ lead to a dominant weight under the dotted action of some Weyl group element.

Example 3.1. We consider the situation where $\dim S = 3$ and the dimension of E is large. We will write $\lambda = (\lambda_1, \lambda_2, \lambda_3, 0, \dots, 0)$ so that $\beta = (-\lambda_3, -\lambda_2, -\lambda_1, 0, \dots, 0)$. After repeated applications of Bott's Theorem, we find the following eight cases to what σ can be. (We write σ as a signed permutation.)

a) $\sigma = \mathrm{id} = (1, 2, \dots, u)$:
$\lambda = (0, 0, \dots, 0)$, $\quad \sigma^\bullet(\beta) = (0, \dots, 0)$, $\quad l(\sigma) = 0$

b) $\sigma = (1, 2, -3, 4, \dots, u)$:
$\lambda = (2u-4, 0, \dots, 0)$, $\quad \sigma^\bullet(\beta) = (0, \dots, 0)$, $\quad l(\sigma) = 2u-5$

c) $\sigma = (1, 3, -2, 4, \dots, u)$:
$\lambda = (2u-3, 1, 0, \dots, 0)$, $\quad \sigma^\bullet(\beta) = (0, \dots, 0)$, $\quad l(\sigma) = 2u-4$

d) $\sigma = (2, 3, -1, 4, \ldots, u)$: 3 subcases each with $l(\sigma) = 2u - 3$:
$\lambda = (\lambda_1, 0, \ldots, 0)$ where $\lambda_1 \geq 2u - 1$ $\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, 1, 1, 0, \ldots, 0)$
$\lambda = (\lambda_1, 1, 0 \ldots, 0)$ where $\lambda_1 \geq 2u - 1$ $\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, 1, 0, \ldots, 0)$
$\lambda = (\lambda_1, 1, 1, 0, \ldots, 0)$ where $\lambda_1 \geq 2u - 2$ $\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, 0, 0 \ldots, 0)$

e) $\sigma = (1, -3, -2, 4, \ldots, u)$:
$\lambda = (2u - 3, 2u - 3, \ldots, 0)$, $\sigma^\bullet(\beta) = (0, \ldots, 0)$, $l(\sigma) = 4u - 9$

f) $\sigma = (2, -3, -1, 4, \ldots, u)$: 3 subcases, each with $l(\sigma) = 4u - 8$:
$\lambda = (\lambda_1, 2u - 2, \ldots, 0)$ where $\lambda_1 \geq 2u - 1$ $\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, 1, 1, 0, \ldots, 0)$
$\lambda = (\lambda_1, 2u - 3, 0 \ldots, 0)$ where $\lambda_1 \geq 2u - 1$ $\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, 1, 0, \ldots, 0)$
$\lambda = (\lambda_1, 2u - 3, 1, 0, \ldots, 0)$ where $\lambda_1 \geq 2u - 2$ $\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, 0, 0 \ldots, 0)$

g) $\sigma = (3, -2, -1, 4, \ldots, u)$: 3 subcases, each with $l(\sigma) = 4u - 7$:
$\lambda = (\lambda_1, \lambda_2, 0 \ldots, 0)$ where $\lambda_1 \geq \lambda_2 \geq 2u$
$\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, \lambda_2 - 2u + 2, 2, 0, \ldots, 0)$
$\lambda = (\lambda_1, \lambda_2, 1, 0 \ldots, 0)$ where $\lambda_1 \geq \lambda_2 \geq 2u - 1$
$\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, \lambda_2 - 2u + 2, 1, 0, \ldots, 0)$
$\lambda = (\lambda_1, \lambda_2, 2, 0, \ldots, 0)$ where $\lambda_1 \geq \lambda_2 \geq 2u - 2$
$\sigma^\bullet(\beta) = (\lambda_1 - 2u + 2, \lambda_2 - 2u + 2, 0 \ldots, 0)$

h) $\sigma = (-3, -2, -1, 4, \ldots, u)$:
$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, 0)$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 2u - 2$ $l(\sigma) = 6u - 12$

Lemma 3.2. *Suppose β is a weight of $\mathrm{Sp}(F)$ such that $\beta = (-\lambda_s, \ldots, -\lambda_2, -\lambda_1, 0, \ldots, 0)$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition.*

a) *Then the only elements σ of the Weyl group for which $\sigma^\bullet(\beta)$ is a dominant weight are parametrized by subsets $U \subseteq \{1, 2, \ldots, s\}$.*
b) *In cycle notation:*

$$\sigma_U = \prod_{k \in U} (k, k + 1 \ldots u)\sigma_0(s, s + 1 \ldots u)^{-1} \tag{10}$$

where the $k \in U$ in the product are listed in decreasing order.
In signed permutation notation, we write $U = \{a_1, \ldots, a_t\}$ listed in increasing order and its complement in $\{1, 2, \ldots, s\}$ as $U' = \{b_1, \ldots, b_{s-t}\}$ also in increasing order. Then

$$\sigma_U = (b_1, \ldots, b_{s-t}, -a_t, -a_{t-1}, \ldots, -a_1, s + 1, \ldots, u) \tag{11}$$

c) $l(\sigma_U) = (2u - s + 1)|U| - \sum_{k \in U} k$.

Proof. We will prove a) and b) together and use the cycle format of Weyl group elements for the proof as it will give us a reduced expression of transpositions. Given a weight β of $\mathrm{Sp}(F)$ we can follow Bott's Algorithm to determine if there exists $\sigma \in \mathcal{W}_u$ such that $\sigma^\bullet(\beta)$ is dominant. For the symplectic case, the algorithm says:

i) to apply to β, $\sigma_0^\bullet$ if $\beta_u < 0$;
ii) to apply to β, the dotted action of the transposition $(j, j + 1)$ where $j = \max\{1 \leq j \leq u : \beta_j < \beta_{j+1}\}$;
iii) repeat these two steps in this order until either the dotted action of any transposition $(j, j+1)$ or σ_0 keeps the weight fixed (in which case β does not map to a dominant weight under the dotted action of any Weyl group element) or until we do obtain a dominant weight.

Since $\beta = (-\lambda_s, \ldots, -\lambda_2, -\lambda_1, 0, \ldots, 0)$, the algorithm shows that if the absolute value of any entry of $\rho + \beta$ is equal to the absolute value of any other, or if one of the entries is zero, then there exists $\sigma \in \mathcal{W}_u$ such that $\sigma^\bullet(\beta) = \beta$. Otherwise, in order to obtain the associated dominant weight, we move $\rho_s - \lambda_1$ to the right end, switch the sign to + if necessary and then reorder. We repeat the process with $\rho_{s-1} - \lambda_2$ and successive terms until we obtain a partition.

In more detail, we begin the algorithm by taking the successive dotted actions of the transpositions $(s, s+1), (s+1, s+2), \ldots, (u-1, u)$. However, if $1 \leq u - s + 1 - \lambda_1 \leq u - s$, say $u - s + 1 - \lambda_1 = k$, then $\sigma^\bullet(\beta) = \beta$ where $\sigma = (s, s+1, \ldots, u-k+1)^{-1}$. Furthermore, if $u - s + 1 - \lambda_1 = 0$ then the dotted action of $\sigma_0(s, s+1, \ldots, u)^{-1}$ fixes β and if $u - s + 1 - \lambda_1 = -k$ where $1 \leq k \leq u - s$ then $(u-k, \ldots, u)\sigma_0(s, \ldots, u)-1$ fixes β.

However, as long $\lambda_1 \leq 2u - 2s + 2$ then there exist choices of λ_i such that the dotted action of $(k, k+1, \ldots, u)\sigma_0(s, s+1, \ldots, u)^{-1}$ where $1 \leq k \leq s$ maps β to a dominant weight. In particular, for $1 \leq k \leq s$ the dotted action of $(k, k+1, \ldots, u)\sigma_0(s, s+1, \ldots, u)^{-1}$ sends

$$\beta = (\underbrace{0, \ldots, 0}_{k-1}, \underbrace{-1, \ldots, -1}_{s-k}, -(2u + 2 - s - k), 0, \ldots, 0) \tag{12}$$

to the dominant weight $(0, 0, \ldots, 0)$. We also note that if $\lambda_1 > 2u - s + 1$ then $(12\ldots, u)\sigma_0(s, s+1, \ldots, u)^{-1}$ maps $(0, \ldots, 0, -\lambda_1, 0, \ldots, 0)$ to the dominant weight $(\lambda_1 - 2u + s - 1, 1^{(s-1)}, 0, \ldots, 0)$.

From the above discussion, we observe that any σ that maps a weight β (as described in the conditions of the lemma) to a dominant weight must be a product of permutations of the form $\sigma_k = (k, k+1, \ldots, u)\sigma_0(s, s+1, \ldots, u)^{-1}$. Hence we may assume from now on that $\sigma = \sigma_{K_c} \cdots \sigma_{K_2}\sigma_{K_1}$. (For the sake of conceptualization, notice that σ_k applied to a weight w takes the negative of the s'th entry of w, places it in the k'th position of w and shifts down the k'th through the $s-1$'st entry of w.)

We now prove part a) and b) together by induction on c. We have already established the basis step and the induction assumption says that there there exists a β of the desired form such that $(\sigma_{K_{c-1}} \cdots \sigma_{K_2}\sigma_{K_1})^\bullet(\beta)$ is a dominant weight if and only if $1 \leq K_1 < K_2 < \cdots < K_{c-1} \leq s$.

If $K_c \leq K_{c-1}$, then the K_c'th entry of $\sigma_{K_c} \cdots \sigma_{K_2}\sigma_{K_1}(\rho + \beta)$ is bigger than the K_{c-1}'th. Thus, $-(u - s + (c-1) - \lambda_{c-1}) \leq -(u - s + c - \lambda_c)$. Thus $1 + \lambda_{c-1} \leq \lambda_c \Leftrightarrow \lambda_{c-1} < \lambda_c$ which is a contradiction since λ is a partition. Thus $K_c > K_{c-1}$.

We can now assume that $1 \leq K_1 < K_2 < \cdots < K_c \leq s$. By induction, there exists a weight β of desired form such that $(\sigma_{K_{c-1}} \cdots \sigma_{K_2}\sigma_{K_1})^\bullet(\beta)$ is dominant. Then $\sigma_{K_{c-1}} \cdots \sigma_{K_2}\sigma_{K_1}$ acting in the standard fashion on $\rho + \beta$ takes the negative of the entries $u - s + 1 - \lambda_1, u - s + 2 - \lambda_2, \ldots, u - s + (c-1) - \lambda_{c-1}$ and then moves them to the K_1'th,K_2'th,$\ldots$, K_{c-1}'th places respectively in the resulting partition. Using β, we must create a β' such that $(\sigma_{K_c} \cdots \sigma_{K_2}\sigma_{K_1})^\bullet(\beta')$ is dominant.

Let $\beta'_i = \beta_i$ for $s - c + 2 \leq i \leq s$. In other words, since β of our desired form corresponds to a partition λ of length s, $\lambda'_i = \lambda_i$ for $1 \leq i \leq c-1$. We then choose λ'_c so that $-(u - s + c - \lambda'_c) = ((\sigma_{K_{c-1}} \ldots \sigma_{K_1})(\rho + \beta))_{K_c - 1} - 1$. Then finally, we set $\beta'_i = \beta_i - 1$ for $K_c - c + 1 \leq i \leq s - c$. This construction gives us a β' that maps to a dominant weight but writing $\beta' = (-\lambda'_s, \ldots, -\lambda'_1, 0, \ldots, 0)$, we must still check that $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is a partition.

It is clear that $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{c-1}$ by induction. Furthermore, since $\beta'_i = \beta_i$ for $K_c - c + 1 \leq i \leq s - c$, then $\lambda'_j = \lambda_j + 1$ for $c + 1 \leq j \leq s + c - K_c$. Thus $\lambda'_{c+1} \geq \ldots \geq \lambda'_s$. Furthermore, it's not too hard to notice that $\lambda'_c \geq u - s + c + 1 \geq \lambda'_{c+1}$. Thus we only need to check that $\lambda'_{c-1} \geq \lambda'_c$ but this is easy to see since by construction, $-(u - s + c - \lambda'_c) \leq -(u - s + c - 1 - \lambda'_{c-1}) - 1 \Leftrightarrow \lambda'_c \leq \lambda'_{c-1}$.

Part c) of the lemma now becomes simple once we remark that Bott's Algorithm ensures that a product of such σ_k is a reduced expression in the Weyl group.

$$\begin{aligned} l(\sigma_k) &= l\big((k, k+1, \ldots, u)\big) + l(\sigma_0) + l\big((s, s+1, \ldots, u)^{-1}\big) \\ &= (2u - s + 1) - k \end{aligned}$$

Therefore $l(\sigma_U) = (2u - s + 1)|U| - \sum_{k \in U} k$.

□

In our efforts to calculate the complex $F_\bullet$ of graded free $k[\mathrm{Hom}(E,F)]$-modules as defined in section 2 (see Theorem 2.6 in particular), Lemma 3.2 indicates that the complex possesses strands parametrized by the set of subsets of $U \subset \{1,2,\ldots,s\}$. In order to understand the structure of the terms in $F_\bullet$, we must analyze how the signed permutations σ_U act on the weights via the dotted action.

Along with the description of σ_U in equation (11), the following three definitions introduce the primary tools which enter into the calculations in the rest of our paper. Furthermore, we have been careful to present the definitions as to ensure that they remain unchanged when we pass to the even and odd orthogonal cases.

Definition 3.3. Let U be a subset of $\{1,2,\ldots,s\}$.

a) Define

$$n(U) = n(\sigma_U) = \begin{cases} \sigma_U(1) - 1 & \text{if } |U| < s \\ s & \text{if } |U| = s \end{cases} \tag{13}$$

$$= \begin{cases} \max\left\{1 \le i \le s \,:\, \{1,\ldots,i\} \subset U\right\} & \text{if } 1 \in U \\ 0 & \text{if } 1 \notin U\ . \end{cases}$$

Intuitively, $n(\sigma_U)$ counts the number of entries $\rho+\beta$ of the form $u-i+1-\lambda$ that get sent to the front under the standard action fo σ_U on weights.

b) Define also

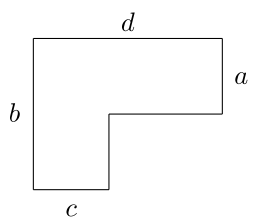
$$t(U) = t(\sigma_U) = |U| = \#\{1 \le i \le s \,:\, \sigma_U(i) < 0\}\ .$$

Definition 3.4. For a given integer $s \le u$, we denote by ω_s the shortest element in the Weyl group $\mathcal{W}_u$ that $\omega_s(i) = -(s+1-i)$ for $1 \le s \le u$.

In particular, when F is symplectic we have $\omega_s = \sigma_{\{1,\ldots,s\}}$ or $\omega_s = (-s,\ldots,-2,-1,s+1,\ldots,u)$ as a signed permutation. We also remark that if U' is the complement of U in $\{1,2,\ldots,s\}$, then $\omega_s\sigma_U = \sigma_U\omega_s = \sigma_{U'}$.

Definition 3.5. Consider the dominant weights of a classical finite dimensional Lie algebra. Let $D_{a,b;c,d}$ be the set of such dominant weights λ of length b with $\lambda_i \le d$ for $1 \le i \le a$ and $\lambda_i \le c$ for $a+1 \le i \le b$. If $a=0$ or $c=0$ we define $D_{a,b;c,d}$ as the set $\{(0,0,\ldots,0)\}$. (If we work with F symplectic or F orthogonal with odd dimension, then dominant weights are simply partitions of integers.)

The set of partitions $D_{a,b;c,d}$ plays a central role in what follows as we use it to describe the possible dominant weights μ such that $K_\lambda E \otimes V_\mu F$ appears as a term in the $F_\bullet$ resolution. (Recall that we write $V_\mu F$ for the irreducble representation of $\mathrm{Sp}(F)$ or $\mathrm{O}(F)$ of highest weight μ.) When the dominant weights are partitions (i.e. in the symplectic and odd orthogonal cases) we may picture $D_{a,b;c,d}$ as the partitions inside the fat hook:



Proposition 3.6. *Let F be a symplectic vector space with* $\dim F = 2u$. *Let D be the set of weights of* $\mathrm{Sp}(F)$ *of type* $\beta = (-\lambda_s, \dots, -\lambda_2, -\lambda_1, 0, \dots, 0)$ *where* $\lambda = (\lambda_1, \dots, \lambda_s)$ *is a partition of length s such that there exists $\sigma \in \mathcal{W}$ such that $\sigma^\bullet(\beta)$ is a dominant weight. Then:*

a) *D is partitioned into subsets D_U where $U \subset \{1, 2, \dots, s\}$ such that for all $\beta \in D_U$, $\sigma_U^\bullet(\beta)$ is a dominant weight.*
b) *Writing $n = n(\sigma_U)$ and $t = t(\sigma_U)$, the set D_U is parametrized by $\mu \in D_{n,s;n,\infty}$ and the defining partitions λ are given by*

$$\lambda_i = \begin{cases} 2u - (s-i) + \sigma_U(s+1-i) + (\mu_{|\sigma_U(s+1-i)|} + 1) & \text{for } 1 \le i \le t \\ -(s-i) + \sigma_U(s+1-i) - (\mu_{|\sigma_U(s+1-i)|} + 1) & \text{for } t+1 \le i \le s \end{cases} \tag{14}$$

or

$$\lambda_{s+1-i} = \begin{cases} 2u + 2 - i + \sigma_U(i) + \mu_{|\sigma_U(i)|} & \text{for } s-t+1 \le i \le s \\ -i + \sigma_U(i) - \mu_{|\sigma_U(i)|} & \text{for } 1 \le i \le s-t \end{cases}. \tag{15}$$

c) *With the above parametrization of D_U, $\sigma_U^\bullet(\beta) = \mu$.*

Before we begin the proof of Proposition 3.6, we make two remarks. First, in the case where $n(\sigma_U) = 0$ (i.e. when $\lambda_1 \le 2u - s - 1$), D_U contains a single element β_0 calculated by the formula (14) with $\mu = 0$ and $\sigma_U(\beta_0) = (0, 0, \dots, 0)$.

Second, from formula (14) it isn't immediately clear that the resulting λ is a partition. We notice that σ_U is positive and increasing over the interval $1 \le i \le s-t$ and negative increasing over $s-t+1 \le i \le s$, and we will use equation (15) to show that λ_{s+1-i} is a nondecreasing sequence of non-negative integers:

- For $1 \le i \le s-t$, $\sigma_U(i)$ increases strictly and hence $\sigma_U(i) - i$ is nondecreasing. But μ is a partition so $\mu_{|\sigma_U(i)|}$ is nonincreasing and hence $-\mu_{|\sigma_U(i)|}$ is also nondecreasing.
- For $s-t+1 \le i \le s$, again since $\sigma_U(i)$ is increasing strictly over this interval, $2u+2-i+\sigma_U(i)$ is nondecreasing. On the other hand, $|\sigma_U(i)| = -\sigma_U(i)$ so this time, $\mu_{|\sigma_U(i)|}$ is nondecreasing.
- By formula (15), $\lambda_s = -1 + \sigma_U(1) - \mu_{|\sigma_U(i)|}$ but by construction $\sigma_U(1) = n+1$ and since $\mu \in D_{n,s;n,\infty}$, $\mu_{n+1} \le n$ and hence $\lambda_s = n - \mu_{n+1} \ge 0$.
- Finally, $\lambda_t - \lambda_{t+1} = 2u + 1 + \sigma_U(s-t+1) - \sigma_U(s-t) + \mu_{|\sigma_U(s-t+1)|} + \mu_{|\sigma_U(s-t)|}$. But $\sigma_U(s-t+1) \ge -s$ and $\sigma_U(s-t) \le s$ and since $s \le u$ we have $\lambda_t - \lambda_{t+1} \ge 1 + \mu_{|\sigma_U(s-t+1)|} + \mu_{|\sigma_U(s-t)|}$.

Putting these four facts together, we notice that λ_{s+1-i} is non decreasing, or in other words that λ is indeed a partition.

Proof. Part a) follows immediately from Lemma 3.2 and the statement in Bott's Theorem that there exists a unique element in the Weyl group that maps β to a dominant under the dotted action.

Part b): We assume the $\beta \in D_U$ where U is a fixed subset of $\{1, 2, \dots, s\}$.

To simplify the notation in the proof, we define the bijection τ on $\{1, 2, \dots, s\}$ by $\tau(i) = |\sigma_U(s + 1 - i)|$. We will use this function during the proof and rephrase our conclusions back in terms of σ_U. We utilize $\tau(i)$ for intuitive purposes because the usual action σ_U sends the $(s+1-i)$'th entry of $\rho + \beta$ to the $\tau(i)$'th entry of the resulting weight.

Now for symplectic group, the half-sum of positive roots is $\rho = (u, u-1, \dots, 1)$. In other words, $\rho_i = u + 1 - i$. Thus we have $(\rho + \beta)_i = u + 1 - i - \lambda_{s+1-i}$ and also $(\rho + \beta)_{s+1-i} = u - s + i - \lambda_i$. Furthermore, for $1 \le i \le t$ the $\tau(i)$'th entry of $\sigma_U(\rho + \beta)$ gets multiplied by -1 so is $-(u - s + i - \lambda_i)$ and hence the $\tau(i)$'th entry of $\sigma_U^\bullet(\beta)$ is $-(u-s+i-\lambda_i) - (u+1-\tau(i)) = \lambda_i - (2u-s+1) + (\tau(i) - i)$. Thus, in $\sigma_U(\rho + \beta)$ and for $1 \le i \le t-1$, any expression containing λ_i always appears before (to the left of) λ_{i+1}.

Now if $n = n(\sigma_U) = 0$, then the entry $-(u-s+1-\lambda_1) \leq -\lambda_s$ and consequently $u-s+1+\lambda_s > \lambda_1$. Since $n = 0$, $\sigma_U^\bullet$ doesn't move the first entry of β. A priori, this means that $\lambda_s \leq s$ and hence $u+1 > \lambda_1$. We conclude that all entries of $\rho+\beta$ are in the interval $[-u, u]$ and thus we must have $\sigma_U(\rho+\beta) = \rho$ and hence $\sigma_U^\bullet(\beta) = 0$.

In fact, n counts how many entries of $\rho+\beta$ the Weyl element σ_U moves to the left of $(\rho+\beta)_1$. If $n = s$, i.e. if $\sigma_U = (-s, -(s-1), \dots, -2, -1, s+1, \dots, u)$, then $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s \geq 2u-s+1$ is the only relevant condition and it satifies the conclusion of Part b) of the proposition. Hence, we'll assume that $n < s$. Since σ_U moves all $(\rho+\beta)_{s+1-i}$ to the left for $1 \leq i \leq n$, we have $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 2u-s+1+n-\lambda_s$.

For $n+1 \leq i \leq s$, σ_U maps the entries of $(\rho+\beta)_{s+1-i}$ to the interval $[u-s+1, u]$. Since a given subset U corresponds to only one Weyl element σ_U, the action of σ_U preserves the order of the $(\rho+\beta)_{s+1-i}$ in the interval $[u-s+1, u]$. We then see that $\sigma_U^\bullet(\beta)$ will be of the form

$$\sigma_U^\bullet(\beta) = (\lambda_1 - 2u + s - 1, \dots, \lambda_n - 2u + s - 1, \nu_1, \dots, \nu_{s-n}, 0, \dots, 0)$$

where ν is a partition in the rectangle R with $s-n$ rows and n columns.

Now for $n+1 \leq i \leq t$, σ_U maps the entry $u-s+i-\lambda_i$ of $\rho+\beta$ to $u-\tau(i)+1+\nu_{\tau(i)-n}$. Hence

$$\begin{aligned} -(u-s+i-\lambda_i) &= u-\tau(i)+1+\nu_{\tau(i)-n} \qquad \iff \\ \lambda_i &= 2u-s+i-\tau(i)+\nu_{\tau(i)-n}+1 \end{aligned}$$

On the other hand, for $t+1 \leq i \leq s$, σ_U maps the entry $u-s+i-\lambda_i$ of $\rho+\beta$ to $u-\tau(i)+1+\nu_{\tau(i)-n}$. Hence

$$\begin{aligned} u-s+i-\lambda_i &= u-\tau(i)+1+\nu_{\tau(i)-n} \qquad \iff \\ \lambda_i &= -s+i+\tau(i)-\nu_{\tau(i)-n}-1 \end{aligned}$$

Gathering the information so far, we have

$$\begin{cases} \lambda_1 \geq \lambda \geq \dots \lambda_n \geq 2u-s+1+n-\lambda_s & \text{for } 1 \leq i \leq n \\ \lambda_i = 2u-s+i-\tau(i)+1+\nu_{\tau(i)-n} & \text{for } n+1 \leq i \leq t \\ \lambda_i = -s+i+\tau(i)-1-\nu_{\tau(i)-n} & \text{for } t+1 \leq i \leq s \end{cases}$$

However, we note that $\lambda_s = \tau(s)-1-\nu_{\tau(s)-n} = n-\nu_1$ since whenever $n < s$, $\tau(s) = \sigma_U(1) = n+1$. In other words, $\lambda_1 \geq \dots \geq \lambda_n \geq 2u-s+1+\nu_1$. Since $\tau(i) = -\sigma_U(s+1-i) = i$ for $1 \leq i \leq n$, we consolidate the three separate conditions by setting $\mu_i = \lambda_i - (2u-s+1)$ for $1 \leq i \leq n$ and $\mu_i = \nu_{i-n}$ for $n+1 \leq i \leq s$. The conditions on λ can now be rephrased by requiring that $\mu \in D_{n,s;n,\infty}$. Recalling that $\tau(i) = |\sigma_U(s+1-i)|$ we obtain equation (14).

Finally, (15) follows from (14) by replacing i with $s+1-i$ everywhere.

Part c) follows immediately from our description of the elements in D_U.

□

Since by definition, $\beta \in D_U$ is of the form $(-\lambda_s, \dots, -\lambda_2, -\lambda_1, 0, \dots, 0)$, we define D'_U as the set of corresponding partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$.

From Proposition 3.6, and in particular formula (14), we notice that for a given $U \subset \{1, 2, \dots, s\}$, there is a distinguished partition $\lambda \in D'_U$ obtained by taking $\mu = 0$. This distinguished partition need not be minimal or maximal by inclusion since for $t+1 \leq i \leq s$, we must subtract $\mu_{|\sigma_U(s+1-i)|}$ from λ_i of the distinguished partition.

As we determine which weights $\beta = (-\lambda_s, \dots, -\lambda_2, -\lambda_1, 0, \dots, 0)$ lead to a dominant weight under the dotted action in the Weyl group of the orthogonal group $O(F)$, we will not repeat the relevant proofs in in their entirety since only a few details change from Proposition 3.6.

3.2. **Some combinatorics of the Weyl group of** $\mathrm{O}(F)$**,** F **of odd dimension.** If $\dim F = f = 2u+1$, the Weyl group for $O(F)$ is given by $\mathcal{W}_u = \mathbb{Z}_2^u \rtimes \mathcal{S}_u$, the weight space is a lattice of rank u and the action of $\mathcal{W}_u$ is identical to the symplectic case.

The first difference between the symplectic case and the odd orthogonal case comes from the fact that the half sum of positive roots ρ now has a different expression. Indeed, if $\dim F = 2u+1$, then $\rho = (u - \frac{1}{2}, u - \frac{3}{2}, \dots, \frac{1}{2})$. Consequently, the dotted action $\sigma^\bullet(\beta) = \sigma(\rho+\beta) - \rho$ will foreseeably differ from the symplectic case. Nonetheless, we still have the following lemma.

Lemma 3.7. *Let* F *be an orthogonal space of dimension* $2u+1$*. Suppose* β *is a weight of* $O(F)$ *such that* $\beta = (-\lambda_s, \dots, -\lambda_2, -\lambda_1, 0, \dots, 0)$ *where* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ *is a partition.*

a) *Then the only elements* σ *of the Weyl group for which* $\sigma^\bullet(\beta)$ *is a dominant weight are parametrized by subsets* $U \subseteq \{1, 2, \dots, s\}$*.*
b) *In signed permutation notation, we write* $U = \{a_1, \dots, a_t\}$ *listed in increasing order and its complement in* $\{1, 2, \dots, s\}$ *as* $U' = \{b_1, \dots, b_{s-t}\}$ *also in increasing order. Then*

$$\sigma_U = (b_1, \dots, b_{s-t}, -a_t, -a_{t-1}, \dots, -a_1, s+1, \dots, u)\ . \tag{16}$$

c) $l(\sigma_U) = (2u - s + 1)|U| - \sum_{k \in U} k$*.*

Proof. The proof of this lemma follows exactly along the same lines of Lemma 3.2. The only difference in the proof comes from ρ but the result remains the same. □

The key combinatorial proposition for the odd orthogonal (B_n) case differes only slightly from the corresponding proposition in the symplectic case. (The formula for λ only differs by 1 for λ_i where $1 \le i \le t$.)

Proposition 3.8. *Let* F *be an orthogonal vector space with* $\dim F = 2u+1$*. Let* D *be the set of weights of* $\mathrm{O}(F)$ *of type* $\beta = (-\lambda_s, \dots, -\lambda_2, -\lambda_1, 0, \dots, 0)$ *where* $\lambda = (\lambda_1, \dots, \lambda_s)$ *is a partition of length* s *such that there exists* $\sigma \in \mathcal{W}$ *such that* $\sigma^\bullet(\beta)$ *is a dominant weight. Then:*

a) D *is partitioned into subsets* D_U *where* $U \subset \{1, 2, \dots, s\}$ *such that for all* $\beta \in D_U$*,* $\sigma_U^\bullet(\beta)$ *is a dominant weight.*
b) *Writing* $n = n(\sigma_U)$ *and* $t = t(\sigma_U)$*, the set* D_U *is parametrized by* $\mu \in D_{n,s;n,\infty}$ *and the defining partitions* λ *are given by*

$$\lambda_i = \begin{cases} 2u - (s-i) + \sigma_U(s+1-i) + \mu_{|\sigma_U(s+1-i)|} & \text{for } 1 \le i \le t \\ \quad -(s-i) - 1 + \sigma_U(s+1-i) - \mu_{|\sigma_U(s+1-i)|} & \text{for } t+1 \le i \le s \end{cases} \tag{17}$$

or

$$\lambda_{s+1-i} = \begin{cases} 2u + 1 - i + \sigma_U(i) + \mu_{|\sigma_U(i)|} & \text{for } s-t+1 \le i \le s \\ \quad -i + \sigma_U(i) - \mu_{|\sigma_U(i)|} & \text{for } 1 \le i \le s-t \end{cases}. \tag{18}$$

c) *With the above parametrization of* D_U*,* $\sigma_U^\bullet(\beta) = \mu$*.*

Proof. The proof is similar to that of Proposition 3.6 with the only change that $\rho = (u - \frac{1}{2}, u - \frac{3}{2}, \dots, \frac{1}{2})$ when F is an orthogonal space of dimension $2u+1$. □

3.3. **Some combinatorics of the Weyl group of** $\mathrm{O}(F)$**,** F **of even dimension.** If $\dim F = f = 2u$, the Weyl group for $O(F)$ is given by $\mathcal{W}_u = \mathbb{Z}_2^{u-1} \rtimes \mathcal{S}_u$, the weight space is a lattice of rank u and the action of $\mathcal{W}_u$ is now different from the symplectic and odd orthogonal cases. The Weyl group is generated by S_u, the symmetric group on u elements, and the element σ_0 that acts on the element $(t_1, t_2, \dots, t_u)$ in the weight space as follows:

$$\sigma_0(t_1, t_2, \dots, t_{u-1}, t_u) = (t_1, t_2, \dots, -t_u, -t_{u-1})$$

Furthermore, we can describe the dominant weight chamber by the following inequalities : $\Delta = \{(\lambda_1, \ldots, \lambda_u) \; : \; |\lambda_1 \geq \ldots \geq \lambda_{u-1} \geq |\lambda_u|\}$. We also need the half sum of positive roots, $\rho = (u-1, u-2, \ldots, 0)$.

We attempt to adapt Lemmas 3.2 and 3.7 in the symplectic and odd orthogonal cases to the even orthogonal case. These lemmas in the first two cases were identical because the Weyl group and the dominant chamber had identical descriptions. In the even orthogonal case, the Weyl group and the dominant weight chamber are no longer the same so we should expect differences to crop up. However, as it turns out, the only difference arises when $s = u$.

Lemma 3.9. *Let F be an orthogonal space of dimension $2u$ and let s be an integer with $s < u$. Suppose β is a weight of $O(F)$ such that $\beta = (-\lambda_s, \ldots, -\lambda_2, -\lambda_1, 0, \ldots, 0)$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition.*

a) *Then the only elements σ of the Weyl group for which $\sigma^\bullet(\beta)$ is a dominant weight are parametrized by subsets $U \subseteq \{1, 2, \ldots, s\}$.*
b) *In cycle notation:*

$$\sigma_U = \prod_{k \in U} (k, k+1 \ldots u-1)\sigma_0(s, s+1 \ldots u)^{-1} \tag{19}$$

where the $k \in U$ in the product are listed in decreasing order.
In signed permutation notation, we write $U = \{a_1, \ldots, a_t\}$ listed in increasing order and its complement in $\{1, 2, \ldots, s\}$ as $U' = \{b_1, \ldots, b_{s-t}\}$ also in increasing order. Then

$$\sigma_U = (b_1, \ldots, b_{s-t}, -a_t, -a_{t-1}, \ldots, -a_1, s+1, \ldots, u-1, (-1)^s u) \; . \tag{20}$$

c) $l(\sigma_U) = (2u - s)|U| - \sum_{k \in U} k$.

Proof. The proof of parts a) and b) of this lemma follows exactly along the same lines of Lemma 3.2 so we only comment on the correct modifications and let the reader fill in the gaps.

In the odd orthogonal case, Bott's algorithm becomes:

i) to apply to β, $\sigma_0^\bullet$ if $\beta_u + \beta_{u-1} < 0$;
ii) to apply to β, the dotted action of the transposition $(j, j+1)$ where where $j = \max\{1 \leq j \leq u \; : \; \beta_j < \beta_{j+1}\}$;
iii) repeat these two steps in this order until either the dotted action of any transposition $(j, j+1)$ or σ_0 keeps the weight fixed (in which case β does not map to a dominant weight under the dotted action of any Weyl group element) or until we do obtain a dominant weight.

The fact that $s < u$ guarantees that the action of $\sigma_0^\bullet$ only changes one negative sign to a positive sign at a time. We could then work through the steps of the algorithm to obtain the lemma.

As for the difference in the formula provided in part c), it is easy to see that this is the proper formula by consulting the reduced cycle notation:

$$l(\sigma_U) = \sum_{k \in U} \big[(u - 1 - k) + 1 + (u - s)\big] = (2u - s)|U| - \sum_{k \in U} k \; .$$

□

Lemma 3.10. *Let F be an orthogonal space of dimension $2u$ and let $s = u$. Suppose β is a weight of $\mathrm{O}(F)$ such that $\beta = (-\lambda_s, \ldots, -\lambda_2, -\lambda_1)$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition.*

a) *Then the only elements σ of the Weyl group for which $\sigma^\bullet(\beta)$ is a dominant weight are parametrized by subsets $U \subseteq \{1, 2, \ldots, s\}$ such that $|U|$ is even. We call $P_2(\{1, 2, \ldots, s\})$ this set of subsets of $\{1, 2, \ldots, s\}$ of even order.*

b) *In signed permutation notation, we write* $U = \{a_1, \dots, a_t\} \in P_2(\{1, 2, \dots, s\})$ *with entries listed in increasing order and its complement in* $\{1, 2, \dots, s\}$ *as* $U' = \{b_1, \dots, b_{s-t-1}\}$ *also in increasing order. Then*

$$\sigma_U = (b_1, \dots, b_{s-t-1}, -a_t, -a_{t-1}, \dots, -a_1) \tag{21}$$

In cycle notation, if $U = \{a_1, \dots, a_t\}$ *with* $|U| = t = 2t'$ *then*

$$\sigma_U = \prod_{1 \le i \le t'} (a_{2i} \dots s)(a_{2i-1} \dots s-1)\sigma_0 \tag{22}$$

c) $l(\sigma_U) = s|U| - \sum_{k \in U} k$.

Proof. To prove parts a) and b), we refer to the Lemma 3.9 for the description of Bott's algorithm for the (D_n) case, i.e. for calculating cohomology of vector bundles over $\mathbb{IG}(s, F)$ where F is orthogonal of even dimension.

Since β is non-increasing, $\rho + \beta$ is a decreasing sequence of integers. Following through Bott's algorithm, we find that we must begin by applying σ_0 if $\beta_s + \beta_{s-1} < 0$ and then apply the proper $\sigma \in \mathcal{W}$ that reorders the entries of $\sigma_0(\rho + \beta)$. But $\sigma_0(\rho + \beta) = ((\rho + \beta)_1, \dots, -(\rho + \beta)_s, -(\rho + \beta)_{s-1})$ and reordering will possibly involve shifting the last two entries.

The set U parametrizes the indices to which a Weyl group element sends the entries of $\rho + \beta$. In particular, since $\rho + \beta$ is already a decreasing sequence, in order for a $\sigma \in \mathcal{W}$ to produce a dominant weight $\sigma^\bullet(\beta)$, we simply need to know which entries of $\rho + \beta$ receive minus signs and to where σ reorders them.

Let us assume that $\beta = (-\lambda_s, \dots, -\lambda_1)$ is such that there exists $\sigma \in \mathcal{W}$ with $\sigma^\bullet(\beta)$ a dominant weight. By Bott's theorem, this element σ is unique. Suppose also that only the entries $(\rho + \beta)_{s+1-i}$ for $1 \le i \le t$ are multiplied by -1 during Bott's algorithm. We now define $U = \{a_1, a_2, \dots, a_t\}$ (where we list the elements in increasing order) to be the subset of $\{1, \dots, s\}$ such that $\sigma(\rho + \beta)_{a_i} = -(\rho + \beta)_{s+1-i}$ for $1 \le i \le t$.

However, the signed permutations $\sigma \in \mathcal{W}$ are such that an even number of entries of σ have negative signs. Hence, the Weyl group of $\mathrm{SO}(F)$ imposes that the subsets $U \subset \{1, 2, \dots, s\}$ we can consider have an even number of elements. This proves parts a) and b).

To prove part c) of the lemma, we use the cycle notation of σ_U and calculate:

$$\begin{aligned} l(\sigma_U) &= \sum_{1 \le i \le t'} \left[(s - a_{2i}) + (s - 1 - a_{2i-1}) + 1\right] \\ &= \sum_{1 \le i \le t} s - a_i \\ &= s|U| - \sum_{k \in U} k \end{aligned}$$

□

Note that we could also parametrize the σ_U such that there exists $\beta = (-\lambda_s, \dots, -\lambda_1)$ that together give a dominant weight $\sigma_U^\bullet(\beta)$ by subsets $U \subset \{1, 2, \dots, s-1\}$. We simply set up a bijection f between $P(\{1, 2, \dots, s\})$ and $P_2(\{1, 2, \dots, s\})$ via:

$$U \longmapsto \begin{cases} U & \text{if } |U| \text{ is even} \\ U \cup \{s\} & \text{if } |U| \text{ is odd} \end{cases}.$$

However, we use the parametrizing set $P_2(\{1, 2, \dots, s\})$ and retain the original definitions for $n(U)$ and $t(U) = |U|$.

As in the cases where F is a symplectic vector space or orthogonal with odd dimension, we must determine which partitions λ that give weights $\beta = (-\lambda_s, \dots, -\lambda_1, 0, \dots, 0)$ that map to a dominant weight under the dotted action of a Weyl group element. In the case when F is an orthogonal vector space of even dimension, as with Lemmas 3.9 and 3.10, we will need to distinguish between the cases when $s < u$ and $s = u$.

Proposition 3.11. *Let F be an orthogonal vector space with $\dim F = 2u$ and suppose that for the parameter s we impose $s < u$. Let D be the set of weights of $O(F)$ of type*

$$\beta = (-\lambda_s, \dots, -\lambda_2, -\lambda_1, 0, \dots, 0)$$

where $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of length s such that there exists $\sigma \in \mathcal{W}$ such that $\sigma^\bullet(\beta)$ is a dominant weight. Then:

a) *D is partitioned into subsets D_U where $U \subset \{1, 2, \dots, s\}$ such that for all $\beta \in D_U$, $\sigma_U^\bullet(\beta)$ is a dominant weight.*

b) *Writing $n = n(\sigma_U)$ and $t = t(\sigma_U)$, the set D_U is parametrized by $\mu \in D_{n,s;n,\infty}$ and the defining partitions λ are given by*

$$\lambda_i = \begin{cases} 2u - (s-i) + \sigma_U(s+1-i) - 1 + \mu_{|\sigma_U(s+1-i)|} & \text{for } 1 \le i \le t \\ -(s-i) + \sigma_U(s+1-i) - 1 - \mu_{|\sigma_U(s+1-i)|} & \text{for } t+1 \le i \le s \end{cases} \tag{23}$$

or

$$\lambda_{s+1-i} = \begin{cases} 2u - i + \sigma_U(i) + \mu_{|\sigma_U(i)|} & \text{for } s-t+1 \le i \le s \\ -i + \sigma_U(i) - \mu_{|\sigma_U(i)|} & \text{for } 1 \le i \le s-t \end{cases} \tag{24}$$

c) *With the above parametrization of D_U, $\sigma_U^\bullet(\beta) = \mu$.*

Proof. The proof is again similar to that of Proposition 3.6 or Proposition 3.8 with the only change that $\rho = (u-1, u-2, \dots, 0)$ or in other words, $\rho_i = u - i$. For any $U \subset \{1, 2, \dots, s\}$, the action of the Weyl group element σ_U as defined in Lemma 3.9 acts effectively in the same way as the Weyl group element that enters into the proof of Propositions 3.6 and 3.8. □

Proposition 3.12. *Let F be an orthogonal vector space with $\dim F = 2u$ and suppose that for the parameter s we impose $s = u$. Let D be the set of weights of $O(F)$ of type $\beta = (-\lambda_s, \dots, -\lambda_1)$ where $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of length s such that there exists $\sigma \in \mathcal{W}$ such that $\sigma^\bullet(\beta)$ is a dominant weight. Then:*

a) *D is partitioned into subsets D_U where $U \in P_2(\{1, 2, \dots, s\})$ such that for all $\beta \in D_U$, $\sigma_U^\bullet(\beta)$ is a dominant weight.*

b) *Writing $n = n(\sigma_U)$ and $t = t(\sigma_U)$, the set D_U is parametrized by $\mu \in D_{n,s;n,\infty}$ and the defining partitions λ are given by*

$$\lambda_i = \begin{cases} 2u - (s-i) + \sigma_U(s+1-i) - 1 + \mu_{|\sigma_U(s+1-i)|} & \text{for } 1 \le i \le t \\ -(s-i) + \sigma_U(s+1-i) - 1 - \mu_{|\sigma_U(s+1-i)|} & \text{for } t+1 \le i \le s \end{cases} \tag{25}$$

or

$$\lambda_{s+1-i} = \begin{cases} 2u - i + \sigma_U(i) + \mu_{|\sigma_U(i)|} & \text{for } s-t+1 \le i \le s \\ -i + \sigma_U(i) - \mu_{|\sigma_U(i)|} & \text{for } 1 \le i \le s-t \end{cases}. \tag{26}$$

c) *With the above parametrization of D_U, $\sigma_U^\bullet(\beta) = \mu$.*

Before begining the proof, we remark that this proposition differs from Proposition 3.11 first in that we use a different parametrizing set to partition D. Secondly, we remind the reader of our definition for the set of dominant weights $D_{a,b;c,d}$. It is the set of dominant weights μ of length b with $\mu_i \le d$ for $1 \le i \le a$ and $\mu_i \le c$ for $a + 1 \le i \le b$. Only when $u = s$ in the even orthogonal case are the elements of $D_{n,s;n,\infty}$ not always partitions. Indeed, for $\mu \in D_{n,s;n,\infty}$, instead of having $\mu_1 \geq \ldots \geq \mu_s \geq 0$ along with the other state inequalities, we have this time $\mu_1 \geq \ldots \geq \mu_{s-1} \geq |\mu_s|$. In particular, μ_s may be negative.

Proof. The proof of this proposition differs significantly enough from the corresponding versions for symplectic and odd orthogonal cases that we repeat the proof and change where appropriate. In this case, not only does the Weyl group change but the difference in the geometry of the dominant chamber also comes into play.

Part a) follows from Lemma 3.10 and the fact that for $\beta \in D$, there exists a unique $\sigma \in \mathcal{W}$ such that $\sigma^{\bullet}(\beta)$ is dominant.

For part b), we assume that $\beta \in D_U$ where U is a fixed subset of $\{1, 2, \ldots, s\}$ of even order. We set $|U| = t$. As in the Propositions 3.6 and 3.8, we define the bijection τ on $\{1, 2, \ldots, s\}$ by $\tau(i) = |\sigma_U(s+1-i)|$ where σ_U is defined in Lemma 3.10 this time. Again, this $\tau(i)$ is defined in such a way that

$$(\sigma(\rho+\beta))_{\tau(i)} = \begin{cases} -(\rho+\beta)_{s+1-i} & \text{if } 1 \leq i \leq t \\ (\rho+\beta)_{s+1-i} & \text{if } t+1 \leq i \leq s \end{cases}$$

Thus, for $1 \leq i \leq s$ we have:

$$\begin{aligned}(\sigma_U^{\bullet}(\beta))_{\tau(i)} &= \begin{cases} -(u-s-1+i-\lambda_i) - (u-\tau(i)) & \text{if } 1 \leq i \leq t \\ (u-s-1+i-\lambda_i) - (u-\tau(i)) & \text{if } t+1 \leq i \leq s \end{cases} \\ &= \begin{cases} \lambda_i - 2u + s + 1 - i + \tau(i) & \text{if } 1 \leq i \leq t \\ -\lambda_i - s - 1 + i + \tau(i) & \text{if } t+1 \leq i \leq s \end{cases}\end{aligned}$$

If $U = \emptyset$, then it is easy to calculate that $D_U = \{(0, 0, \ldots, 0)\}$, $\sigma_U = id$ and that $\sigma_U^{\bullet}(\beta) = (0, 0, \ldots, 0)$.

Now if $U \neq \emptyset$ and $n(U) = n(\sigma_U) = 0$, then $-(\rho+\beta)_s < (\rho+\beta)_1$ and all entries of $\rho+\beta$ are in $[-(u-1), u-1]$. It is not hard to check with a few simple cases that having required $\sigma_U(\rho+\beta)-\rho$ be a dominant weight, we must have $(\rho+\beta)_1 = u-1$ and hence that $\lambda_s = 0$. Furthermore, $(\sigma_U^{\bullet}(\beta))_1 = 0$ and since $\sigma_U^{\bullet}(\beta)$ is a dominant weight for $SO(F)$ with F even dimensional then $\sigma_U^{\bullet}(\beta) = (0, 0, \ldots, 0)$.

Assume $n = s$ which can only happen when $U = \{1, 2, \ldots, s\}$ and s is even. Then $\sigma_U = (-s, -(s-1), \ldots, -1)$ and since $\sigma_U^{\bullet}(\beta)$ is a dominant weight, we have

$$\lambda_1 \geq \ldots \geq \lambda_{s-1} \geq |\lambda_s - 2u + s + 1| + 2u - s - 1 \geq 2u - s - 1 .$$

This is the only condition on λ.

Assume now that $n < s$. Then since σ_U moves all the entries $(\rho+\beta)_{s+1-i}$ to the left of $(\rho+\beta)_1$, we have $\lambda_n - u + s + 1 - n > u - 1 - \lambda_s$ and hence $\lambda_1 \geq \ldots \geq \lambda_n \geq 2u - s - 1 + n - \lambda_s$. For $n+1 \leq i \leq s$, the Weyl element σ_U maps the entries of $(\rho+\beta)_{s+1-i}$ into the interval $[1, s]$ and so we have

$$\sigma_U^{\bullet}(\beta) = (\lambda_1 - 2u + s + 1, \ldots, \lambda_n - 2u + s + 1, \nu_1, \ldots, \nu_{s-n})$$

where ν satisfies the following relation $n \geq \nu_1 \geq \ldots \geq \nu_{s-n-1} \geq |\nu_{s-n}|$.

Using the assumption that $n < s$ we have $\tau(s) = \sigma_U(1) = n+1$ and $\lambda_s = n - \nu_1$. Collecting information so far, we have

$$\begin{cases} \lambda_1 \geq \ldots \geq \lambda_n \geq 2u - s - 1 + \nu_1 & \text{for } 1 \leq i \leq n \\ \lambda_i = 2u - s - 1 + i - \tau(i) + \nu_{\tau(i)-n} & \text{for } n+1 \leq i \leq t \\ \lambda_i = -s - 1 + i + \tau(i) - \nu_{\tau(i)-n} & \text{for } t+1 \leq i \leq s \end{cases}$$

where ν satisfies the condition $n \geq \nu_1 \geq \ldots \geq \nu_{s-n-1} \geq |\nu_{s-n}|$.

Similar to the previous versions of this proposition for the (C_n) and the (B_n) cases, we claim we can rewrite this condition on λ by

$$\lambda_i = \begin{cases} 2u - s - 1 + i - \tau(i) + \mu_{\tau(i)} & \text{for } 1 \le i \le t \\ -s - 1 + i + \tau(i) - \mu_{\tau(i)} & \text{for } t+1 \le i \le s \end{cases}$$

where $\mu \in D_{n,s;n,\infty}$ where $n = n(U)$.

The only possible obstruction to this claim is whether the fact that λ is a partition imposes a lower bound restriction on μ_s that is more stringent than $\mu_s \ge -\mu_{s-1}$. To prove the claim, we check that this obstruction does not occur with two cases.

- Suppose $s \in U$. Then $\tau(t) = s$ and since λ is a partition, $\lambda_t \ge \lambda_{t+1}$. Hence $2u - s - 1 + t - \tau(t) + \mu_{\tau(t)} \ge -s-1+t+1+\tau(t+1) - \mu_{\tau(t+1)}$ and therefore $\mu_s \ge -s+1+\tau(t+1) - \mu_{\tau(t+1)}$. However, $\tau(t+1) \le s-1$ so $\mu_{\tau(t+1)} \ge \mu_{s-1}$ and thus
$$\tau(t+1) - s + 1 - \mu_{\tau(t+1)} \le -\mu_{\tau(t+1)} \le -\mu_{s-1}$$
We then conclude that if $s \in U$ then for any $\mu \in D_{n,s;n,\infty}$ our definition of λ does indeed produce a partition.
- Suppose now that $s \notin U$. Then $\tau(t+1) = s$ and since $\lambda_t \ge \lambda_{t+1}$, this time we get $\mu_s \ge -s + 1 + \tau(t) - \mu_{\tau(t)}$. Again $\tau(t) \le s-1$ so $\mu_{\tau(t)} \ge \mu_{s-1}$. Thus
$$\tau(t) - s + 1 - \mu_{\tau(t)} \le -\mu_{\tau(t)} \le -\mu_{s-1}$$
Again, if $s \notin U$ then for any $\mu \in D_{n,s;n,\infty}$ our definition of λ does indeed produce a partition.

These two statements together prove our claim and our claim settles parts b) and c). □

With all these combinatorial results, we are now in a position to prove a theorem that provides a relatively simple combinatorial description for the terms in the $F_\bullet$-complex which in turn allows us to answer algebraic-geometric questions about special orbits.

4. The algebraic geometry of special orbits $\bar{O}_{r_1,r_2}$

4.1. **The $F_\bullet$ complex for special orbits.** Before stating the main theorem of this section, let us remind the reader of a few definitions.

In section 2, we defined the parameter

$$\varepsilon = \begin{cases} 1 & \text{if } F \text{ is an orthogonal space} \\ -1 & \text{if } F \text{ is a symplectic space} \end{cases} .$$

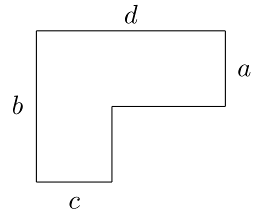
This parameter allows us to describe the $F_\bullet$ complex for symplectic and orthogonal groups on the vector space F in a uniform manner.

Also for any signed permutation $\sigma \in Z_2^u \rtimes S_u$, define its action on a sequence of length u of numbers $l = (l_1, l_2, \ldots, l_u)$ as follows:

$$\sigma(l) = (\text{sign}(\sigma(1))l_{\sigma(1)}, \ldots, \text{sign}(\sigma(u))l_{\sigma(u)}) .$$

Finally, in section 3, we defined the set of dominant weights $D_{a,b;c,d}$ for any classical finite dimensional Lie algebra as follows. Let $D_{a,b;c,d}$ be the set of dominant weights λ of length b with $\lambda_i \le d$ for $1 \le i \le a$ and $\lambda_i \le c$ for $a+1 \le i \le b$. If $a = 0$ or $c = 0$ we define $D_{a,b;c,d}$ as the set $\{(0,0,\ldots,0)\}$.

When we work with F symplectic or F orthogonal with odd dimension, then dominant weights are simply partitions of integers. In that case, we may picture $D_{a,b;c,d}$ as the partitions inside the fat hook:



Lemma 4.1. *Let λ be a partition and β the weight given by $(-\lambda_s, \dots, -\lambda_2, -\lambda_1, 0, \dots, 0)$. The cohomology group $H^j(\mathbb{IG}(s, F), K_{\lambda'}E \otimes K_\beta \mathcal{S}) = 0$ if $l(\lambda') > e$ where $e = \dim E$. Consequently, when looking for non-vanishing cohomology, in each set of weights D_U (as defined in Propositions 3.6, 3.8, 3.11 and 3.12), we only need to consider the weights β determined by the partition $\mu \in D_{n,s;n,d}$ where $d = e - (f - s - \varepsilon)$.*

Proof. The first part of this lemma follows immediately from equation (8). For the second part, note that if we look for non-vanishing cohomology, we only consider weights λ with $l(\lambda') = \lambda_1 \leq e$. By comparing formulas (14), (17), (23) and (25), we see that whether F is symplectic or orthogonal, two situations can occur:

a) $n = 0$ and the lemma holds true trivially since in this case $D_{n,s;n,d}$ consists of the zero partition.
b) $n > 0$, in which case $\sigma_U(s) = -1$ and $\lambda_1 = f - s - \varepsilon + \mu_1$. Hence, we must have $\mu_1 \leq e - (f - s - \varepsilon)$.

□

Theorem 4.2. *Let E be a vector space of dimension e, F a vector space of dimension f that is symplectic or orthogonal. Consider the $F_\bullet$ complex associated to $\xi = E \otimes \mathcal{S}$ where $\mathcal{S}$ is the tautological bundle over the isotropic Grassmannian $\mathbb{IG}(s, F)$, where $s \leq u = \lfloor \frac{f}{2} \rfloor$. Assume that $e > f - 2s$. Then the following hold:*

a) *If $e \leq f - s - \varepsilon$, the $F_\bullet$ complex consists only of terms $K_\lambda E \otimes A(-|\lambda|)$. Furthermore, $F_\bullet$ is a resolution indexed by a partition λ from an $(e \times s)$-box such that if $\lambda = (a_1, \dots, a_r | b_1, \dots, b_r)$ then $b_i = a_i + (f - 2s - \varepsilon)$ for $1 \leq i \leq r$. Furthermore, the term $K_\lambda E \otimes A(-|\lambda|)$ appears in degree $d(\lambda)$ where:*

$$d(\lambda) = \begin{cases} \sum_{i=1}^r a_i & \text{if } F \text{ is symplectic} \\ \sum_{i=1}^r (a_i - 1) & \text{if } F \text{ is orthogonal} \end{cases}. \tag{27}$$

(Let us call $P(e, s, m)$ the set of partitions $\lambda = (a_1, \dots, a_r | b_1, \dots, b_r)$ in an $(e \times s)$-box such that $b_i = a_i + m$).

b) *If $e > f - s - \varepsilon$, the term $K_\lambda E \otimes V_\mu F \otimes A(-|\lambda|)$ appears in $F_\bullet$ if there exists a partition $\nu \in P(e, s, f - 2s - \varepsilon)$ such that $\mu \in D_{n,s;n,e-f+s+\varepsilon}$, where $n = \#\{i : \nu_i = s\}$ and*

$$\lambda' = \nu' + \sigma_\nu(\mu)$$

where we define the signed permutation σ_ν as follows.

Write $\nu = (a_1, \dots, a_r | b_1, \dots, b_r)$. Let $U = \{s + 1 - a_i : 1 \leq i \leq r\}$ or $U = \{s + 1 - a_i : 1 \leq i \leq r\} \cup \{s\}$ only if F is orthogonal of even dimension, $s = f/2$ and r is odd. For $U = \{u_1, \dots, u_t\}$ (listed in increasing order) a subset of $\{1, 2, \dots, s\}$ with complement $U' = \{v_1, \dots, v_{s-t}\}$ (listed in increasing order), define $\sigma_\nu = (u_1, \dots, u_t, -v_{s-t}, \dots, -v_1, s + 1, \dots, u)$.

Furthermore, the term $K_\lambda E \otimes V_\mu F \otimes A(-|\lambda|)$ appears in degree

$$d(\lambda) = d(\nu) + |\lambda| - |\nu| = d(\nu) + \sum_{i=1}^{s} (\sigma_\nu(\mu))_i$$

where $d(\nu)$ is calculated according to the formula in part a).

We remark that in part a) of Theorem 4.2, the complex $F_\bullet$ is a specialization of a resolution of Pfaffians.

Before we begin the proof of the theorem, we prove a combinatorial lemma that comes in handy.

Lemma 4.3. *Let $P(\{1,\dots,s\})$ be the power set of $\{1,\dots,s\}$ and let us consider the map f from $P(\{1,\dots,s\})$ to the set of partitions defined by $f(U) = (u_t - t, u_{t-1} - (t-1), \dots, u_1 - 1)$ where the elements of $U = \{u_1,\dots,u_t\}$ are listed in increasing order. Let U' be the complement of U in $\{1,\dots,s\}$. If $|U| = t$, then $f(U)$ is a partition in a rectangle $t \times (s-t)$ and $f(U)' = \widetilde{f(U')}$ where $\widetilde{f(U')}$ is the complementary partition to $f(U')$ in the $(s-t) \times t$ rectangle.*

Proof. It is a standard exercise to see that partitions in a $t \times (s-t)$-rectangle, paths from $(0,0)$ to $(s-t,t)$ in the grid of this rectangle and subsets of $\{1,2,\dots,s\}$ are all equivalent objects from a combinatorial point of view. The bijection between subsets $U = \{u_1,\dots,u_t\}$ and paths from $(0,0)$ to $(s-t,t)$ is given by taking the points in the grid $(u_i - i, i)$ for $1 \le i \le t$ to be points immediately before which the path is vertical. We create a unique path in this way. Explicitly, the path is:

$$(0,0) \to (u_1-1,0) \to (u_1-1,1) \to (u_2-2,1) \to (u_2-2,2) \to \dots \to (u_t-t,t) \to (s-t,t)\ . \tag{28}$$

To obtain the path corresponding to $f(U)'$, we must replace each point (x,y) in the path with $(t-y, s-t-x)$ and reverse the result. Then it's easy to see that to obtain the path for $\widetilde{f(U)}'$, we would simply replace each point (x,y) in that path (28) with (y,x). We get the path:

$$(0,0) \to (0,u_1-1) \to (1,u_1-1) \to (1,u_2-2) \to (2,u_2-2) \to \dots \to (t,u_t-t) \to (t,s-t)\ .$$

Now, we can eliminate extraneous points from the path describing $\widetilde{f(U)}'$ as follows. If $u_{i+1} - (i+1) = u_i - i$, we eliminate the sequence $\to (i, u_i - i) \to (i, u_{i+1} - (i+1)) \to$. The only remaining $i \in \{1,\dots,t\}$ will be precisely where $u_{i+1} - (i+1) > u_i - i$. It's now easy to see that such indicies correpond precisely to the sequence $v_j - j$ for $1 \le j \le s-t$ where $U' = \{v_1,\dots,v_{s-t}\}$. □

Proof of Theorem 4.2. The first claim in part a) simply summarizes the content of Lemma 4.1 which was written to include all three cases: F symplectic, odd orthogonal or even orthogonal.

To prove the rest of part a) we will first calculate the modified complex $F_\bullet^\dagger$ obtained by assuming that $\dim E = f - s - \varepsilon$. Representation theory then tells us that if $\dim E = e < f - s - \varepsilon$ we obtain $F_\bullet$ from $F_\bullet^\dagger$ by setting equal to 0 any term $K_\lambda E$ where $l(\lambda) = \lambda'_1 > \dim E$.

According to Propositions 3.6, 3.8, 3.11 and 3.12, since $e \le f - s - \varepsilon$ the terms in $F_\bullet^\dagger$ are parametrized by subsets $U \subset \{1,2,\dots,s\}$.

We write the subset $U \subset \{1,2,\dots,s\}$ as $U = \{u_1,u_2,\dots,u_t\}$ and the complement $U' = \{v_1,\dots,v_{s-t}\}$ with elements listed in increasing order. We notice from the afore mentioned propositions that $K_\lambda E$ occurs in $F_\bullet^\dagger$ when:

$$\lambda'_i = \begin{cases} f - s + i - u_i - \varepsilon & \text{for } 1 \le i \le t \\ -s + i + v_{s+1-i} - 1 & \text{for } t+1 \le i \le s\ . \end{cases} \tag{29}$$

As it turns out, we can prove part b) of the theorem for F symplectic, F odd orthogonal and F even orthogonal with $s < \frac{1}{2}f$ using exactly the same reasoning. Hence, let us assume for the moment that we are in one of these three cases.

Let us determine the Frobenius notation for the partition λ. We assume that we can write $\lambda = (a_1, \ldots, a_r | b_1, \ldots, b_r)$ and hence $\lambda' = (b_1, \ldots, b_r | a_1, \ldots, a_r)$. Note that since $u_i \leq s$, $f - s + i - u_i - \varepsilon \geq i$ for $i \leq t$ (this fails when F even orthogonal with $s = \frac{1}{2}f$), but also $-s + i + v_{s+1-i} - 1 < i$ for $i \geq t+1$. Consequently, we see that in the Frobenius description of λ we must have $r = t$.

It is easy to see that for $1 \leq i \leq t$, $b_i = \lambda'_i - i + 1 = f - s + 1 - \varepsilon - u_i$. However, we also claim that $a_i = s + 1 - u_i$ for $1 \leq i \leq t$. Indeed, notice that $a_i = \lambda_i - i + 1$ for $1 \leq i \leq t$. Hence, we need to calculate λ_j for $1 \leq j \leq t$. But $\lambda_j = \#\{i : \lambda'_i \geq j\}$. We also know that $\lambda'_j \geq j$ so for $1 \leq j \leq t$,

$$\begin{aligned}
\lambda_j &= t + \#\{t+1 \leq i \leq s : \lambda'_i \geq j\} \\
&= t + \#\{t+1 \leq i \leq s : -s + i + v_{s+1-i} - 1 \geq j\} \\
&= t + \#\{1 \leq i' \leq s - t : v_{i'} - i' \leq j\} && \text{where } i' = s + 1 - i \\
&= t + (s - t - (u_j - j)) && \text{by Lemma 4.3} \\
&= s - u_j + j
\end{aligned}$$

Consequently, for $1 \leq i \leq t$, $a_i = \lambda_i - i + 1 = s + 1 - u_i$ and hence we can write $\lambda' = (f + 1 - \varepsilon - s - u_i | s + 1 - u_i)$ for $1 \leq i \leq t$ and in particular we notice the relation $b_i - a_i = f - 2s - \varepsilon$.

Next, we remark here that it is an easy combinatorial exercise to prove that our definition of $\lambda' = (f+1-\varepsilon-s-u_i|s+1-u_i)$ in fact produces a bijection between the power set $P(\{1, 2, \ldots, s\})$ and the set of partitions $\lambda = (a_1, \ldots, a_t | b_1, \ldots, b_t)$ in an $(f-s-\varepsilon)\times s$ rectangle such that $b_i = a_i + f - 2s - \varepsilon$.

To finish proving part a), we now only need to determine the degree $d(\lambda)$ of $F_\bullet$ in which $K_\lambda E \otimes A(-|\lambda|)$ occurs. By Theorem 2.6, the term $K_\lambda E \otimes A(-|\lambda|)$ occurs as a summand of F_i where $i = |\lambda| - l(\sigma_U)$, where σ_U is defined in the Lemmas 3.2, 3.7, 3.9 and 3.10. Using Equation (29), we get:

$$\begin{aligned}
|\lambda| - l(\sigma_U) &= \sum_{i=1}^{t}(f - s + i - u_i - \varepsilon) + \sum_{i=t=1}^{s}(-s + i + v_{s+1-i}) - \left[(f - s + \frac{1-\varepsilon}{2}) - \sum_{i=1}^{t} u_i\right] \\
&= -t\frac{1+\varepsilon}{2} + \sum_{i=1}^{s} i + \sum_{i=t+1}^{s}(-s - 1 + v_{s+1-i}) \\
&= s(s+1) - t\frac{1+\varepsilon}{2} - (s-t)(s+1) - \sum_{i=1}^{t} u_i
\end{aligned}$$

Thus, we find that the term $K_\lambda E \otimes A(-|\lambda|)$ occurs in degree

$$d(\lambda) = \begin{cases} t(s+1) - \sum_{k\in U} k & \text{if } F \text{ is symplectic} \\ ts - \sum_{k\in U} k & \text{if } F \text{ is orthogonal .} \end{cases}$$

With $\lambda = (a_1, \ldots, a_t | b_1, \ldots, b_t)$, we saw that for $1 \leq i \leq t$ we have $a_i = s + 1 - u_i$ where $U = \{u_1, \ldots, u_t\}$. Hence

$$\begin{aligned}
d(\lambda) &= \begin{cases} t(s+1) - \sum_{i=1}^{t}(s + 1 - b_i) & F \text{ symplectic} \\ ts - \sum_{i=1}^{t}(s + 1 - b_i) & F \text{ orthogonal} \end{cases} \\
&= \begin{cases} \sum_{i=1}^{t} b_i & F \text{ symplectic} \\ \sum_{i=1}^{t}(b_i - 1) & F \text{ orthogonal} \end{cases}
\end{aligned}$$

This proves the enunciation of part a) when $e = f - s - \varepsilon$ in all but one case which we treat now.

Now we need to cover the last case, namely when F is orthogonal of even dimension and $s = \frac{1}{2}f$. We remind the reader that the set $P_2(\{1, 2, \ldots, s\})$, the set of even sized subsets of $\{1, 2, \ldots, s\}$, is in bijection with $P(\{1, 2, \ldots, s-1\})$ via $U \mapsto U^\circ = U - \{s\}$. As before, for $t + 1 \leq i \leq s$, we have

$-s+i+v_{s+1-i}-1 < i$ but this time $\lambda'_i = f-s+i-\varepsilon-u_i \geq i$ if and only if $u_i < s$. Hence when we write $\lambda = (a_1,\ldots,a_r|b_1,\ldots,b_r)$, $r = |U|$ if $s \notin U$ and $r = |U|-1$ if $s \in U$. In other words, $r = |U^\circ|$.

Using the same reasoning as above, we still deduce that for $1 \leq i \leq |U^\circ|$ we have $b_i = f-s+1-\varepsilon-u_i = s-u_i$ and $a_i = s+1-u_i$. Hence, since $U^\circ \subset \{1,2,\ldots,s-1\}$, the set of all partitions λ are the partitions in a rectangle $(s-1)\times s$ with $b_i = a_i - 1$. This reproduces precise the statement of part b) of the theorem with $e = f-s-\varepsilon$.

To calculate the degree of the term $K_\lambda E \otimes A(-|\lambda|)$ in the $F_\bullet^\dagger$ complex, we check two cases. First, if $s \notin U$ then $r = |U^\circ| = |U| = t$ and

$$d(\lambda) = st - \sum_{i=1}^{t} u_i = st - \sum_{i=1}^{t}(s+1-b_i) = \sum_{i=1}^{t}(b_i-1) = \sum_{i=1}^{r}(b_i-1)\ .$$

On the other hand, if $s \in U$ then $U^\circ = U - \{s\}$, $r = t-1$ and

$$d(\lambda) = st - \sum_{i=1}^{t} u_i = st - \Big(\sum_{i=1}^{t-1} s+1-b_i\Big) - s = st-(s+1)(t-1)-s+\sum_{i=1}^{t-1} b_i = \sum_{i=1}^{r}(b_i-1)\ .$$

This completes the proof of part b) with $e = f-s-\varepsilon$ in all cases.

Finally, if $e < f-s-\varepsilon$, we simply ignore any term $K_\lambda E \otimes A(-|\lambda|)$ containing a partition λ with $\lambda'_1 > e$. Hence, $K_\lambda E \otimes A(-|\lambda|)$ appears as a term in $F_\bullet$ if and only if $\lambda = (a_1,\ldots,a_r|b_1,\ldots,b_r)$ is a partition in a $(e\times s)$-rectangle such that $b_i = a_i + (f-2s-\varepsilon)$. This concludes the proof of part b) of the theorem.

Part b) of the theorem is essentially a summary of Propositions 3.6, 3.8, 3.11, and 3.12 in which we calculated λ in terms of the parametrizing subset $U \subset \{1,2,\ldots,s\}$ and $\mu \in D_{n(U),s;n(U),\infty}$ on a case by case basis. These propositions, along with Lemma 4.1, say that $K_\lambda E \otimes V_\mu F \otimes A(-|\lambda|)$ appears in the $F_\bullet$ complex if

$$\lambda'_i = \begin{cases} f-s+i-u_i-\varepsilon+\mu_{u_i} & \text{for } 1 \leq i \leq t \\ -s+i+v_{s+1-i}-1-\mu_{v_{(s+1-i)}} & \text{for } t+1 \leq i \leq s \end{cases} \tag{30}$$

and $\mu \in D_{n(U),s;n(U),e-f+s+\varepsilon}$.

Let us then write

$$\nu'_i = \begin{cases} f-s+i-u_i-\varepsilon & \text{for } 1 \leq i \leq t \\ -s+i+v_{s+1-i}-1 & \text{for } t+1 \leq i \leq s \end{cases}$$

for the subset U. Then by the methods of part b) we determine that ν is a partition in an $(e\times s)$-box such that if $\nu = (a_1,\ldots,a_r|b_1,\ldots,b_r)$ in Frobenius notation then $b_i = a_i + (f-2s-\varepsilon)$. Now we can restate (30) as:

$$\lambda' = \nu' + \sigma_\nu(\mu)$$

for some $\mu \in D_{n(U),s;n(U),e-f+s+\varepsilon}$ and where σ_ν is defined in the statement of the theorem.

We remark that following the language of Lemma 3.2 and the definition of Equation 11, we have $\sigma_\nu = \sigma_{U'}$ where U' is the complement of U in $\{1,2,\ldots,s\}$. (Furthermore, for all $U \subset \{1,2,\ldots,s\}$ and for all $1 \leq i \leq s$, $\sigma_{U'}(i) = -\sigma_U(s+1-i)$.) It is also clear by the definition that $n(U) = \#\{i : \nu_1 = s\}$.

Finally, we must calculate the degree $d(\lambda)$ in which the term $K_\lambda E \otimes V_\mu F \otimes A(-|\lambda|)$ appears in the $F_\bullet$ complex. However, σ_U depends only on the partition $\nu \in P(e,s,f-2s-\varepsilon)$ and hence

$$|\lambda| - l(\sigma_U) = |\lambda| - |\nu| + |\nu| - l(\sigma_U) = d(\nu) + |\lambda| - |\nu|$$

where

$$d(\nu) = \begin{cases} \sum_{i=1}^{r} a_i & \text{if } F \text{ is symplectic} \\ \sum_{i=1}^{r}(a_i-1) & \text{if } F \text{ is orthogonal} \end{cases}$$

as in part b) of the theorem.

The second expression for $d(\lambda)$ follows from (30). □

Here is another formula for the degree $d(\lambda)$.

Proposition 4.4. *The term $K_\lambda E \otimes V_\mu F \otimes A(-|\lambda|)$ as defined in Theorem 4.2 appears in the $F_\bullet$ complex in degree*

$$d(\lambda) = \begin{cases} \sum_{i=t+1}^{s} \lambda'_i + \sum_{i=1}^{t} (i + \sigma_\nu(\mu)_i) & \text{if } F \text{ is symplectic} \\ \sum_{i=t+1}^{s} \lambda'_i + \sum_{i=1}^{t} (i - 1 + \sigma_\nu(\mu)_i) & \text{if } F \text{ is orthogonal} \end{cases}$$

where $\lambda' = \nu' + \sigma_\nu(\mu)$ and $t = |U|$.

Proof. We first remark that whether F is symplectic or orthogonal, $\sigma_\nu(\mu)_i = \mu_{u_i}$ for $1 \leq i \leq t$ where $U = \{u_1, \dots, u_t\}$ is described in part c) of the above theorem. In particular, these numbers are nonnegative. We now prove the proposition in two parts.

First, let us consider when F is symplectic. We use formula 30 for λ' and then we have

$$\begin{aligned} d(\lambda) &= |\lambda| - l(\sigma_U) \\ &= \sum_{i=1}^{t} f - s + i - u_i + 1 + \mu_{u_i} + \sum_{i=t+1}^{s} \lambda'_i - \left[(f - s + 1) - \sum_{i=1}^{t} u_i\right] \\ &= \sum_{i=1}^{t} (i + \mu_{u_i}) + \sum_{i=t+1}^{s} \lambda'_i \end{aligned}$$

Second, consider the case when F is orthogonal. Again, we use formula 30 for λ' and then we have

$$\begin{aligned} d(\lambda) &= |\lambda| - l(\sigma_U) \\ &= \sum_{i=1}^{t} f - s + i - u_i - 1 + \mu_{u_i} + \sum_{i=t+1}^{s} \lambda'_i - \left[(f - s) - \sum_{i=1}^{t} u_i\right] \\ &= \sum_{i=1}^{t} (i - 1 + \mu_{u_i}) + \sum_{i=t+1}^{s} \lambda'_i \end{aligned}$$

□

Corollary 4.5. *This degree $d(\lambda)$ is non-negative.*

Proof. When F is symplectic, both the summations in Proposition 4.4 are positive as long as $t > 0$. If $t = 0$ then of course the summations are 0.

When F is orthogonal, both summations in Proposition 4.4 are nonnegative (though not necessarily non-zero). □

4.2. **Properties of special orbits $\bar{O}_{r_1,r_2}$, F symplectic.** We remind the reader that for the purposes of this paper, we call closures of special orbits the varieties $\bar{O}_{r_1,r_2}$ when either $f = 2r_1 - r_2$ or $e = r_1$. We set these apart from the general case because by Proposition 2.9, we can define a desingularization if $f = 2r_1 - r_2$ or a nice surjective morphism $\pi : Z \to \bar{O}_{r_1,r_2}$ if $e = r_1$. In both cases, we employ a bundle ξ as presented in section 2.2 which has the form $\xi = E \otimes \mathcal{S}$, where $\mathcal{S}$ is the tautological vector bundle over the isotropic Grassmannian $\mathbb{IG}(s, F)$.

Having calculated the $F_\bullet$ complex associated to this $\xi = E \otimes \mathcal{S}$, we reduce our geometric questions to simple combinatorial remarks.

Theorem 4.6. *Let E be a vector space of dimension e and F a symplectic vector space of dimension f. If*

a) *$e = r_1$, then $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay and normal.*
b) *$f = 2r_1 - r_2$, then $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay, normal and has rational singularities.*

Proof. For a), if $e = r_1$ and $f > 2r_1 - r_2$ then we are in case c) of Proposition 2.11. Our methods do not provide us with a desingularization so the complex $F_\bullet$ associated to ξ is not necessarily a resolution of the coordinate ring of the orbit closure. However, since $e = r_1$, the orbit closure $\bar{O}_{e,r_2}$ is a specialization of the variety Y_{r_2} of pfaffians of rank r_2 on the matrix of $\mathrm{Sp}(F)$ invariants. Furthermore, by simple inspection, one finds that the complex $F_\bullet$ as constructed in part a) of Theorem 4.2 using $s = \frac{f-r_2}{2}$ produces the same resolution as obtained by Józefiak, Pragacz and Weyman in [8] for Y_{r_2}. Therefore, $F_\bullet$ is indeed a resolution of the coordinate ring $k[\bar{O}_{e,r_2}]$. Since this resolution has the same length as the codimension of $\bar{O}_{e,r_2}$, this variety is Cohen-Macaulay. Furthermore, in [8] the authors prove that Y_{r_2} is normal, $\bar{O}_{r_1,r_2}$ is as well.

For b), if $f = 2r_1 - r_2$ then Proposition 2.9 provides a desingularization of $\bar{O}_{r_1,r_2}$ and hence we can utilize Theorem 2.7 since a desingularization is a birational isomorphism. We have calculated the complex $F_\bullet$. By Corollary 4.5 we know that $d(\lambda) \geq 0$ and by Proposition 4.4 we determine that $d(\lambda) = 0$ if and only if $t = \operatorname{rank} \nu = 0$. If $\nu = 0$ then $n = 0$, $\mu = 0$ and $\lambda = 0$. Consequently, the zero'th term of the complex $F_\bullet$ is

$$F_0 = K_0 E \otimes H^0(\mathbb{IG}(r_1 - r_2, F), K_0 S) \otimes A = H^0(\mathbb{IG}(s, F), \bigwedge^0 \xi) \otimes A = A\,.$$

By Theorem 2.7 we deduce that $\bar{O}_{r_1,r_2}$ is normal, has rational singularities, and that $F_\bullet$ is a finite free resolution of $k[\bar{O}_{r_1,r_2}]$. □

Remark 4.1. Though having rational singularities implies that the special orbit $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay, we can deduce the Cohen-Macaulay property directly by showing that the codimension of $\bar{O}_{r_1,r_2}$ in $\mathrm{Hom}(E, F)$ is equal to the length of the resolution $F_\bullet$. We prove this directly only for the case where $f = 2r_1 - r_2$. With $e = r_2$, the calculations are similar and easier.

Proof of Remark 4.1. Since $f = 2r_1 - r_2$, set $s = f - r_1$, so that we also have $r_2 = f - 2s$.

The dimension of $\bar{O}_{r_1,r_2}$ is equal to the dimension of the desingularization and hence:

$$\begin{aligned}\dim \bar{O}_{r_1,r_2} &= \dim \mathbb{IG}(s, F) + \operatorname{rank} \mathcal{T} \\ &= s(f - s) - \frac{1}{2}s(s-1) + (f - s)e\end{aligned}$$

Thus:

$$\begin{aligned}\operatorname{codim} \bar{O}_{r_1,r_2} &= fe - s(f - s) + \frac{1}{2}s(s-1) - (f - s)e \\ &= s(e - f + s) + \frac{1}{2}s(s-1)\end{aligned}$$

On the other, we need to calculate the length of the $F_\bullet$ resolution. We do this using degree formulas in Theorem 4.2. Again, using Frobenius notation $\nu = (a_1, \dots, a_t | b_a, \dots, b_t)$, the formula is

$$d(\lambda) = d(\nu) + \sum_{i=1}^{s} (\sigma_\nu(\mu))_i = \sum_{i=1}^{t} a_i + \sum_{i=1}^{s} (\sigma_\nu(\mu))_i$$

We need to find the pair (ν, μ) with $\nu \in P(e, s, f - 2s + 1)$ and $\mu \in D_{n,s;n,e-f+s+1}$ where $n = \#\{i : \nu_i = s\}$ that produce the largest $d(\lambda)$. This is easy to do but we must split into two cases: when $e = f - s$ and when $e \geq f - s + 1$.

If $e = f - s$, then by Theorem 4.2 we have $\mu = 0$. Furthermore, the largest we can make $d(\lambda) = d(\nu)$ is with the partition $\nu = ((s-1)^e) = (s-1, \dots, 2, 1 | e, e-1, \dots, e-s+2)$. Thus

$$d(\lambda) = \sum_{i=1}^{s-1} i = \frac{s(s-1)}{2} = \operatorname{codim} \bar{O}_{r_1,r_2} \quad \text{when } e = f-2$$

If $e \geq f - s + 1$ then the largest we can make $d(\lambda)$ is easy to find. The largest possible value for $\sum_{i=1}^{t} a_i$ occurs when $t = s$ and $a_i = s+1-i$. This solution does occur in $P(e, s, f-2s+1)$ precisely when $e \geq f - s + 1$. This corresponds to $\nu = (s^{(f-s+1)})$, $U = \{1, 2, \dots, s\}$ and $n(U) = s$. However, among all possible choices for a subset $U \subset \{1, 2, \dots, s\}$ and $\mu \in D_{n(U),s;n(U),e-f+s+1}$, the largest we can make

$$\sum_{i=1}^{s} \sigma_\nu(\mu)$$

occurs when $U = \{1, 2, \dots, s\}$, $n(U) = s$ and $\mu = ((e-f+s-1)^s)$. Hence, the length of the $F_\bullet$ resolution is

$$\begin{aligned} d(\lambda_{max}) &= \sum_{i=1}^{s} i + s(e-f+s-1) \\ &= \frac{s(s-1)}{2} + s(e-f+s) \\ &= \operatorname{codim} \bar{O}_{f-s,f-2s} \end{aligned}$$

□

4.3. **Examples of Resolutions of Special Orbits when F is Symplectic.** In order to illustrate the results provided so far, we present two examples that produce $F_\bullet$ complexes for $\xi = E \otimes \mathcal{S}$. The first complex is one that is already known whereas the second one is new. Furthermore, in both examples below, we will use $s = \operatorname{rank} \mathcal{S} = 3$ with different dimentions of E and F in order to allow the interested reader to compare the work we do here with the Example in subsection 3.1.

Example 4.7. Let us consider the case where $e = \dim E = 4$, $f = \dim F = 6$, $r_1 = 3$ and $r_2 = 0$. Then since $f = 2r_1 - r_2$ we do indeed have a special orbit with $s = \frac{f-r_2}{2} = 3$. By formula (5) $\bar{O}_{3,0}$ has codimension 6 in $\operatorname{Hom}(E, F)$. In this case, $e = f - s + 1$ so by part a) of Theorem 4.2, the $F_\bullet$ resolution only involves terms of the form $K_\lambda E \otimes A(-|\lambda|)$. Part b) of the theorem tells us that

$$F_i = \bigoplus_{d(\lambda)=i} K_\lambda E \otimes A(-|\lambda|)$$

where $\lambda \in P(4, 3, 1)$.

Consequently, here are all the relevant partitions λ:

λ	$d(\lambda)$
0	0
(4, 1, 1)	3
(3, 1, 0)	2
(2, 0, 0)	1
(4, 4, 2)	5
(4, 3, 1)	4
(3, 3, 0)	3
(4, 4, 4)	6

Calling $A = k[\mathrm{Hom}(E,F)]$, the corresponding $F_\bullet$ resolution is:

$$0 \longrightarrow K_{3,3,3,3}E \otimes A(-12) \longrightarrow K_{3,3,2,2}E \otimes A(-10) \longrightarrow K_{4,3,3,1}E \otimes A(-8) \longrightarrow$$
$$(K_{3,1,1,1}E \oplus K_{2,2,2}E) \otimes A(-6) \longrightarrow K_{2,1,1}E \otimes A(-4) \longrightarrow K_{1,1}E \otimes A(-2) \longrightarrow A$$

We recognize this complex as the Koszul complex of $\bigwedge^2 E$, i.e. the Koszul complex on $\mathrm{Sp}(F)$-invariants in $\mathrm{Hom}(E,F)$.

Example 4.8. Let us consider the case where $e = \dim E = 7$, $f = \dim F = 8$, $r_1 = 5$ and $r_2 = 2$. Once again, since $f = 2r_1 - r_2$, $O_{5,2}$ is a special orbit with $s = 3$. As one might expect, this example involves a much longer resolution but illustrates phenomena that does not happen for Koszul complexes or for smaller values of s. Since $e \geq f - s$, the special orbit corresponding to this data is $\bar{O}_{5,2}$ and by formula (5), $\bar{O}_{5,2}$ has codimension 9 in $\mathrm{Hom}(E,F)$, which in this case has dimension 56.

The parametrizing set of pairs of partitions (ν,μ) is slightly more complicated than in the previous example since $e - (f - s + 1) = 1$. We can call this parametrizing set

$$\Delta = \{(\nu,\mu) \,|\, \nu \in P(7,3,3) \text{ and } \nu \in D_{n,3;n,1} \text{ where } n = \#\{i : \nu_i = 3\}\}$$

The weight μ on F must be in the set

$$D_{n,3;n,1} = \begin{cases} 3 \times 1\text{-box} & \text{if } n > 0 \\ \{0\} & \text{if } n = 0 \end{cases}$$

The following table contains all the data necessary to construct the $F_\bullet$-complex according to our current technique.

ν	n	μ	σ_ν	λ	$d(\lambda)$
0	0	$(0,0,0)$	$(-3,-2,-1,4)$	$(0,0,0)$	0
$(3,1^5)$	1	$(0,0,0)$	$(1,-3,-2,4)$	$(3,1^5)$	3
		$(1,0,0)$	$(1,-3,-2,4)$	$(3,1^6)$	4
		$(1,1,0)$	$(1,-3,-2,4)$	$(2,1^6)$	3
		$(1,1,1)$	$(1,-3,-2,4)$	(1^7)	2
$(2,1^4)$	0	$(0,0,0)$	$(2,-3,-1,4)$	$(2,1^4)$	2
(1^4)	0	$(0,0,0)$	$(3,-2,-1,4)$	(1^4)	1
$(3^2,2^4)$	2	$(0,0,0)$	$(1,2,-3,4)$	$(3^2,2^4)$	5
		$(1,0,0)$	$(1,2,-3,4)$	$(3^2,2^4,1)$	6
		$(1,1,0)$	$(1,2,-3,4)$	$(3^2,2^5)$	7
		$(1,1,1)$	$(1,2,-3,4)$	$(3,2^6)$	6
$(3,2^4,1)$	1	$(0,0,0)$	$(1,3,-2,4)$	$(3,2^4,1)$	4
		$(1,0,0)$	$(1,3,-2,4)$	$(3,2^4,1^2)$	5
		$(1,1,0)$	$(1,3,-2,4)$	$(2^5,1^2)$	4
		$(1,1,1)$	$(1,3,-2,4)$	$(2^6,1)$	5
(2^5)	0	$(0,0,0)$	$(2,3,-1,4)$	(2^5)	3
(3^6)	3	$(0,0,0)$	$(1,2,3,4)$	(3^6)	6
		$(1,0,0)$	$(1,2,3,4)$	$(3^6,1)$	7
		$(1,1,0)$	$(1,2,3,4)$	$(3^6,2)$	8
		$(1,1,1)$	$(1,2,3,4)$	(3^7)	9

Thus, the corresponding complex $F_\bullet$ is the following:

$$
\begin{array}{c}
0\\
\downarrow\\
K_{(3^7)}E\otimes V_{(1^3)}F\otimes A(-21)\\
\downarrow\\
K_{(3^6,2)}E\otimes V_{(1^2)}F\otimes A(-20)\\
\downarrow\\
(K_{(3^6,1)}E\otimes F\otimes A(-19))\oplus(K_{(3^2,2^5)}E\otimes V_{(1^2)}F\otimes A(-16))\\
\downarrow\\
(K_{(3^6)}E\otimes A(-18))\oplus(K_{(3^2,2^4,1)}E\otimes F\otimes A(-15))\oplus(K_{(3,2^5)}E\otimes V_{(1^3)}F\otimes A(-13))\\
\downarrow\\
(K_{(3^2,2^4)}E\otimes A(-14))\oplus(K_{(3,2^4,1^2)}E\otimes F\otimes A(-13))\oplus(K_{(2^6,1)}E\otimes V_{(1^3)}F\otimes A(-13))\\
\downarrow\\
(K_{(3,2^4,1)}E\otimes A(-12))\oplus(K_{(3,1^6)}E\otimes F\otimes A(-9))\oplus(K_{(2^5,1^2)}E\otimes V_{(1^2)}F\otimes A(-12))\\
\downarrow\\
(K_{(3,1^5)}E\otimes A(-8))\oplus(K_{(2,1^6)}E\otimes V_{(1^2)}F\otimes A(-8))\oplus(K_{(2^5)}E\otimes A(-10))\\
\downarrow\\
(K_{(2,1^4)}E\otimes F\otimes A(-6))\oplus(K_{(1^7)}E\otimes V_{(1^3)}F\otimes A(-7))\\
\downarrow\\
K_{(1^4)}E\otimes A(-4)\\
\downarrow\\
A\\
\downarrow\\
0
\end{array}
$$

4.4. **Properties of Special Orbits of $\bar{O}_{r_1,r_2}$ with F Orthogonal of Odd Dimension.** The results we obtain for orthogonal cases turn out not to be quite as pleasant as in the symplectic case. In particular, the closures of special orbits will not necessarily be normal.

Theorem 4.9. *Let E be a vector space of dimension e and F an orthogonal vector space of dimension $f=2u+1$. Two possibilities occur:*

a) *If $e=r_1$ and $2r_1-r_2<f$ then $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay and normal.*
b) *If $f=2r_1-r_2$ then the normalization of $\bar{O}_{r_1,r_2}$ has rational singularities.*

Proof. If $r_1=e$, then as in the symplectic case $\bar{O}_{e,r_2}$ is a specialization of the variety of symmetric matrices of rank at most r_2 of $O(F)$-invariants in $\mathrm{Hom}(E,F)$. Thus, by results of Józefiak, Pragacz and Weyman in [8], the orbit closure $\bar{O}_{e,r_2}$ is normal.

On the other hand, if $f=2r_1-r_2$ we set $s=r_1-r_2=\frac{f-r_2}{2}$. We can utilize the $F_\bullet$-complex since in the language of the geometric technique $\pi':X'_s\longrightarrow\mathrm{Hom}(E,F)$ is a desingularization of $\bar{O}_{f-s,f-2s}$. We need to calculate the F_0 term. Using Proposition 4.4, we see that $d(\lambda)=0$ only if $t=\operatorname{rank}\nu=0$ (in which case, $\nu=0$ and $\lambda=0$) or $t=1$, $\mu_{u_1}=0$ and $\lambda'_i=0$ for $i\geq 2$. Suppose $u_1\neq 1$ then $n=0$ and $\lambda=\nu$; but also if $u_1=1$ then $\mu_1=0$ so $\mu=0$ and hence again $\lambda=\nu$. We then see that $\nu=\lambda=(1^{(f-2s)})=(1^{r_2})$ is the only partition that satisfies $d(\lambda)=0$ and $\operatorname{rank}\nu=1$.

Thus, setting $A = k[\mathrm{Hom}(E, F)]$, we find that

$$F_0 = A \oplus \bigwedge^{r_2} E \otimes A(-r_2)$$

This shows that $\bar{O}_{r_1,r_2}$ might not be normal. However, Corollary 4.5 shows that if $i < 0$ then $F_i = 0$. Consequently, the normalization of $\bar{O}_{r_1,r_2}$ has rational singularities. □

We remark that 4.9 does not conclude one way or the other whether the variety $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay when F is orthogonal with $\dim F = 2r_1 - r_2$.

4.5. **Properties of Special Orbits of $\bar{O}_{r_1,r_2}$ with F Orthogonal of Even Dimension.** We follow the same approach as in the previous section but there turns out to be one more case to consider when F is orthogonal of even dimension.

Theorem 4.10. *Let E be a vector space of dimension e and F an orthogonal vector space of dimension $f = 2u$. Three possibilities occur:*

a) *If $e = r_1$ and $2r_1 - r_2 < f$ then $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay and normal.*
b) *If $f = 2r_1 - r_2$ and $r_2 \neq 0$ then the normalization of $\bar{O}_{r_1,r_2}$ has rational singularities.*
c) *If $r_1 = \frac{f}{2}$ and $r_2 = 0$ (which imply $f = 2r_1 - r_2$) then $\bar{O}_{r_1,r_2}$ is Cohen-Macaulay, normal and has rational singularities.*

Proof. Parts a) and b) are identical to Theorem 4.9 but we must consider part c) separately because of the exception case described in Theorem 4.2.

For part c), we do have $f = 2r_1 - r_2$ and we set $s = u = \frac{f}{2} = r_1$ since $r_2 = 0$. We can utilize the complex $F_\bullet$ since in the language of the geometric technique $\pi' : X'_s \longrightarrow \mathrm{Hom}(E, F)$ is a desingularization of $\bar{O}_{u,0}$. We need to calculate the F_0 term.

By Theorem 4.2, the partitions ν which enter into the description of the complex $F_\bullet$ are in the set $P(e, u, -1)$ which means that if we write ν in Frobenius notation as

$$\nu = (a_1, \ldots, a_r | b_1, \ldots, b_r)$$

then $a_1 \leq u$, $b_1 \leq e$ and $b_i = a_i - 1$ for $a \leq i \leq r$. Again, using Proposition 4.4, we see that $d(\lambda) = 0$ only if $t = \mathrm{rank}\,\nu = 0$ (in which case, $\nu = 0$ and $\lambda = 0$) or $t = 1$, $\mu_{u_1} = 0$ and $\lambda'_i = 0$ for $i \geq 2$. However, by part c) in Theorem 4.2 this latter case cannot occur since when $s = \frac{f}{2}$ the parametrizing subset $U = \{u_1, \ldots, u_t\}$ is such that $t = |U|$ is always even.

Consequently, if $r_1 = \frac{f}{2}$ and $r_2 = 0$ when we set $A = k[\mathrm{Hom}(E, F)]$ we find that $F_0 = A$. By Theorem 2.7, $\bar{O}_{r_1,r_2}$ is normal and has rational singularities. That the orbit closure is Cohen-Macaulay follows as well. □

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